The Niels Henrik Abel mathematics competition: Final 2021–2022

8 March 2022 (English)

In the final round of the Abel contest there are four problems (seven subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

Allowed aids are writing paper, bilingual dictionaries, and writing tools including compass and ruler, but not protractor.

Problem 1

a. Determine all positive integers n such that $2022 + 3^n$ is a perfect square.

b. Find all primes p and positive integers n satisfying

$$n\,5^{n-n/p} = p!\,(p^2+1) + n$$

(note: $p! = 1 \cdot 2 \cdot \dots \cdot p$)

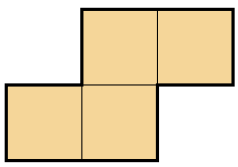
Problem 2

a. A triangle ABC with circumcircle ω satisfies $|AB| > |AC|$. Points X and Y on ω are different from A, such that the line AX passes through the midpoint of BC, AY is perpendicular to BC, and XY is parallel to BC. Find $\angle BAC$.

b. Triangles ABC and DEF have pairwise parallel sides: $EF \parallel BC$, $FD \parallel CA$, and $DE \parallel AB$. The line m_A is the reflection of EF through BC, similarly m_B is the reflection of FD through CA, and m_C the reflection of DE through AB. Assume that the lines m_A, m_B and m_C meet in a common point. What is the ratio between the areas of triangles ABC and DEF?

Problem 3

Nils has an $M \times N$ board where M and N are positive integers, and a tile shaped as shown below. What is the smallest number of squares that Nils must colour, so that it is impossible to place the tile on the board without covering a coloured square? The tile can be freely rotated and mirrored, but it must completely cover four squares.



Problem 4

a. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f\left(\frac{1}{x}\right) \geq 1 - \frac{\sqrt{f(x)f\left(\frac{1}{x}\right)}}{x} \geq x^2 f(x)$$

for all $x \in \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers.

b. Do there exist 2022 polynomials with real coefficients, each of degree equal to 2021, so that the $2021 \cdot 2022 + 1$ coefficients in their product are equal?