The Niels Henrik Abel mathematics competition: Final 2022–2023

7 March 2023 (English)

In the final round of the Abel contest there are four problems (eight subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

Allowed aids are writing paper, bilingual dictionaries, and writing tools including compass and ruler, but not protractor.

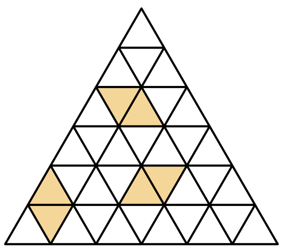
Problem 1

a. In the triangle ABC, X lies on the side BC, Y on the side CA, and Z on the side AB with $YX \parallel AB$, $ZY \parallel BC$ and $XZ \parallel CA$. Show that X, Y, and Z are the midpoints of the respective sides of ABC.

b. In the triangle ABC, points D and E lie on the side BC, with $CE = BD$. M is the midpoint of AD. Show that the centroid of ABC lies on ME.

Problem 2

a. The sides of an equilateral triangle with sides of length n have been divided into equal parts, each of length 1, and lines have been drawn through the points of division parallel to the sides of the triangle, thus dividing the large triangle into many small triangles. Nils has a pile of rhombic tiles, each of side 1 and angles 60° and 120°, and wants to tile most of the triangle using these, so that each tile covers two small triangles with no overlap. In the picture, three tiles are placed somewhat arbitrarily as an illustration. How many tiles can Nils fit inside the triangle?



b. Arne and Berit are playing a game. They have chosen positive integers m and n with $n \geq 4$ and $m \leq 2n + 1$. Arne begins by choosing a number from the set $\{1, 2, \dots, n\}$, and writes it on a blackboard. Then Berit picks another number from the same set, and writes it on the board. They continue alternating turns, always choosing numbers that are not already on the blackboard. When the sum of all the numbers on the board exceeds or equals m, the game is over, and whoever wrote the last number has won. For which combinations of m and n does Arne have a winning strategy?

Problem 3

a. Find all nonnegative integers n, a, and b satisfying

$$2^a + 5^b + 1 = n!\,.$$

b. Find all integers a and b satisfying

$$\begin{aligned} &a^6 + 1 \mid b^{11} - 2023\,b^3 + 40\,b \quad \text{and} \\ &a^4 - 1 \mid b^{10} - 2023\,b^2 - 41. \end{aligned}$$

(Given integers u and v, the notation $u \mid v$ means that there exists some integer c so that $uc = v$.)

Problem 4

a. Assuming a, b, and c are the side lengths of a triangle, show that

$$\frac{a^2 + b^2 - c^2}{ab} + \frac{b^2 + c^2 - a^2}{bc} + \frac{c^2 + a^2 - b^2}{ca} > 2.$$

Also show that the inequality does not necessarily hold if you replace 2 (on the right hand side) by a bigger number,

b. Find all functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$f\big(f(x) + y\big) = f(y) + x \quad \text{for all } x, y \in \mathbb{R}_+.$$

(Here $\mathbb{R}_+$ is the set of all real numbers $x > 0$.)