The Niels Henrik Abel mathematics competition: First round 2023–2024

9 November 2023 (English)

ABEL KONKURRANSEN

Do not turn the page until told to do so!

The first round of the Abel competition consists of 20 multiple choice problems to be solved in 100 minutes. Only one of the five alternatives is correct. Write your answers in the lower left hand side of the form.

You get 5 points for each correct answer, 1 point for a blank answer, and 0 points for a wrong answer. This yields a total between 0 and 100 points. A totally blank response results in 20 points.

No aids other than scratch paper and writing implements (including compass and ruler, but not protractor) are allowed.

When told to, you may turn over the page and begin working on the problems.

Fill in using block letters

Name	Date of birth
Email	Gender F ☐ M ☐
School	Class

☐ Check the box to allow us to put your name on the score board.
(Only applies to the highest scores, approx. top 33 %.)

Answers

1		11	
2		12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	
10		20	

For the teacher

Correct: ☐ $\cdot\, 5 =$ ☐

Unanswered: $+$ ☐

Points: $=$ ☐

Problem 1

What is the smallest positive integer having exactly six (positive) divisors, including 1 and the number itself? For example, 75 has six divisors: 1, 3, 5, 15, 25, and 75.

A 6 **B** 12 **C** 18 **D** 24 **E** 32

Problem 2

Peter tries to find real numbers x and y satisfying $x^2 + y^2 = 5$ and $xy = 1$. What can be said, with certainty, about such numbers?

A $x < y$ **B** $x = y$ **C** $x > y$
D Neither statement **A**, **B**, nor **C** is certain.
E Such real numbers x, y do not exist.

Problem 3

Nils has a box of five differently flavoured chocolate bars: Orange, banana, chili, grape, and apple. He does not think chili sounds good, so he wants to get it over with, and eat it as one of the first two bars. Orange sounds good, so he wants to save it and eat it as one of the last two. In how many different orders could he eat the five bars, satisfying these requirements?

A 4 **B** 8 **C** 18 **D** 24 **E** 30

Problem 4

Henrik makes three isosceles triangles, all having two sides of length 1. The angle between the equal sides is as given below. Which triangle has the largest area?

A 30° **B** 60° **C** 75° **D** 100° **E** 105°

Problem 5

Nils has four identical pairs of shoes. He is going on a trip, and wants to bring two pairs of shoes. He picks four shoes at random. The first two of them are right shoes. What is the probability that the next two will be left shoes, so that the four shoes make up two pairs?

A $\frac{2}{5}$ **B** $\frac{3}{14}$ **C** $\frac{1}{4}$ **D** $\frac{4}{9}$ **E** $\frac{3}{16}$

Problem 6

A positive integer k is such that the final digit of k^8 is 6. What is the final digit of k^4?

A 2 **B** 4 **C** 6 **D** 8 **E** Not uniquely determined.

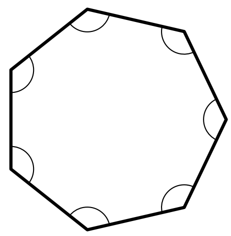
Problem 7

Helene studies languages. She has borrowed ten different books from the library. Five of them are in Polish, and the other five in Ukrainian. For variety, she does not want to read two books in the same language successively. In how many different orders can she read all the books?

A 240 **B** 3125 **C** 10395 **D** 28800 **E** 1814400

Problem 8

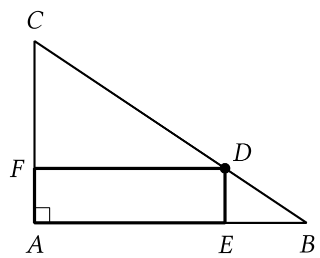
All the sides of a regular polygon are of equal length, and all the angles between neighbouring sides are equal. (The figure shows a regular heptagon.) Which one of these could be an angle in a regular polygon?



A 81° **B** 100° **C** 121° **D** 144° **E** 169°

Problem 9

The right triangle ABC has sides $AB = 6$ and $AC = 4$. Nils wants to place a point D on the hypotenuse BC, and draw the rectangle $DFAE$ with corners E and F on the legs of the triangle, as shown in the picture. If he places D on the hypotenuse so that the rectangle $DFAE$ is half the area of the triangle ABC, how long must CD be?



A 2 **B** 3 **C** $\sqrt{13}$ **D** $2\sqrt{2}$ **E** $\frac{2\sqrt{3}}{3}$

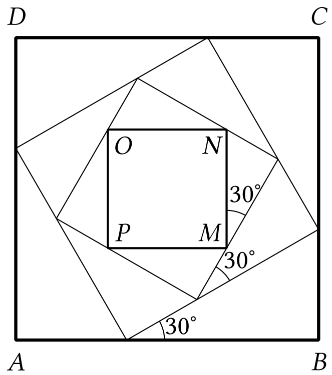
Problem 10

The Gemini school is attended by 160 pupils in total, divided into 80 pairs of twins. One week the school had outdor activities three days in a row. Each day each pupil could choose either to go for a hike or to play football, but twins always had to choose differently. Exactly 20 pupils played football all three days. How many pupils played football exactly two of the three days?

A 10 **B** 20 **C** 30 **D** 40 **E** 60

Problem 11

The square *ABCD* has side length 1, and contains three squares nested inside each other. Each square is rotated 30° in relation to the square outside, and has its corners on the sides of the outside square. What is the side length of the innermost square *MNOP*?



A $\dfrac{2}{5}$ **B** $6\sqrt{3}-10$ **C** $\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}$
D $\dfrac{\sqrt{2}}{4}$ **E** $8\sqrt{2}-11$

Problem 12

Lisa writes down the number 1 on a sheet of paper. She may choose to stop there, or she can choose to either multiply the number by 4 or add 5 to it, before erasing the number on the sheet and replacing it by the new number. She may repeat this for as long as she wants, but she always stops before the number becomes greater than 100. How many different numbers (including 1) can she end up with?

A 40 **B** 41 **C** 42 **D** 43 **E** 44

Problem 13

Which number has the same value as

$$\sqrt{\frac{\pi+\sqrt{\pi^2-\pi}}{2}}+\sqrt{\frac{\pi-\sqrt{\pi^2-\pi}}{2}}\,?$$

A π **B** $\sqrt{\pi}$ **C** $\sqrt{\pi^2-\pi}$ **D** $\pi-\sqrt{\pi}$ **E** $\sqrt{\pi+\sqrt{\pi}}$

Problem 14

The quadrangle $ABCD$ has right angles in the corners B and C, and side lengths $AB = 72$, $BC = 81$, and $CD = 90$. The diagonals intersect in point E. How large is the area of the triangle BCE?

A 405 **B** 810 **C** 1215 **D** 1620 **E** 1780

Problem 15

Which number is the largest?

A $\sqrt{4} - \sqrt{3}$ **B** $\sqrt{3} - \sqrt{2}$ **C** $\sqrt[3]{3} - \sqrt[3]{2}$ **D** $\sqrt[3]{2} - 1$ **E** $\sqrt[4]{2} - 1$

Problem 16

Calculate

$$\frac{(0! + 1!) \cdot (1! + 2!) \cdot (2! + 3!) \cdot \cdots \cdot (2023! + 2024!)}{0! \cdot 1! \cdot 2! \cdot \cdots \cdot 2023!}$$

(Here $n! = 1 \cdot 2 \cdot \cdots \cdot n$, and $0! = 1! = 1$.)

A $2023!$ **B** $2023! \cdot 2023$ **C** $2024!$ **D** $2025!$
E None of these.

Problem 17

The sequence a_n is defined by $a_1 = 23$ and $a_n = 5a_{n-1} + 7$ for $n = 2, 3, \ldots$. What is the penultimate (next to last) digit of a_{2023}?

A 1 **B** 2 **C** 4 **D** 6 **E** 9

Problem 18

In the regular decagon (ten-sided polygon) $ABCDEFGHIJ$ with side length 1 and centre at O, point B' is the mirror image of B through the line segment AC. Then the distance from O to B' is

A 1 **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1+\sqrt{2}}{2}$ **D** $\frac{\sqrt{5}}{2}$ **E** None of these.

Problem 19

Arne, Berit, and Camilla want to reserve a group room tomorrow (between one midnight and the next). It is only possible to reserve the room for a whole number of hours, always starting on a whole hour. Arne wants to reserve the room for three consecutive hours. Berit wants it for four consecutive hours, and Camilla wants is for seven consecutive hours. Each one picks the starting time at random, with equal probability for each possible choice. For example, Arne can pick one of the times 00, 01, 02, ..., 21, but not 22 or 23, since he must finish before midnight.

Example: Arne 14–17, Berit 9–13, Camilla 17–24 is OK.
Example: Arne 15–18, Berit 9–13, Camilla 17–24 yields an overlap A/C.

What is the probability that none of the reservations overlap?

A $\frac{1}{6}$ **B** $\frac{3}{16}$ **C** $\frac{5}{27}$ **D** $\frac{13}{63}$ **E** $\frac{25}{133}$

Problem 20

Jon writes two integers u and v on a sheet of paper. Then he puts $a_0 = 0$, and calculates $a_n = ua_{n-1} + v$ for $n = 1, 2, 3, \ldots, 2023$. Finally, he ends up with $a_{2023} = 2023$. How many possible pairs (u, v) could Jon have started with?

A 1 **B** 2 **C** 3 **D** 4 **E** Infinitely many.