The Niels Henrik Abel mathematics competition: Final 2023–2024

5 March 2024 (English)

In the final round of the Abel contest there are four problems (eight subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

Allowed aids are writing paper, bilingual dictionaries, and writing tools including compass and ruler, but not protractor.

Problem 1

a. Determine all integers $n \geq 2$ such that $n \mid s_n - t_n$ where s_n is the sum of all the integers in the interval $[1, n]$ that are mutually prime to n, and t_n is the sum of the remaining integers in the same interval.

b. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that the numbers

$$n, f(n), f(f(n)), \ldots, f^{m-1}(n)$$

are distinct modulo m for all integers n, m with $m > 1$.

(Here f^k is defined by $f^0(n) = n$ and $f^{k+1}(n) = f(f^k(n))$ for $k \geq 0$.)

Problem 2

a. Positive integers $a_0 < a_1 < \ldots < a_n$, are to be chosen so that $a_j - a_i$ is not a prime for any i, j with $0 \leq i < j \leq n$. For each $n \geq 1$, determine the smallest possible value of a_n.

b. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$xf(f(x) + y) = f(xy) + x^2$$

for all $x, y \in \mathbb{R}$.

Problem 3

a. Determine the smallest constant N so that the following may hold true:

Geostan has deployed secret agents in Combostan. All pairs of agents can communicate, either directly or through other agents. The *distance* between two agents is the smallest number of agents in a communication chain between the two agents.

Andreas and Edvard are among these agents, and Combostan has given Noah the task of determining the distance between Andreas and Edvard. Noah has a list of numbers, one for each agent. The number of an agent describes the longest of the two distances from the agent to Andreas and Edvard. However, Noah does not know which number corresponds to which agent, or which agents have direct contact.

Given this information, he can write down N numbers and prove that the distance between Andreas and Edvard is one of these N numbers. The number N is independent of the agents' communication network.

b. A 2024-*table* is a table with two rows and 2024 columns containg all the numbers 1, 2, ..., 4048. Such a table is *evenly coloured* if exactly half of the numbers in each row, and one number in each column, is coloured red. The *red sum* in an evenly coloured 2024-table is the sum of all the red numbers in the table.

Let N be the largest number such that every 2024-table has an even colouring with red sum $\geq N$. Determine N, and find the number of 2024-tables such that every even colouring of the table has red sum $\leq N$.

Problem 4

a. The triangle ABC with $AB < AC$ has an altitude AD. The points E and A lie on opposite sides of BC, with E on the circumcircle of ABC. Furthermore, $AD = DE$ and $\angle ADO = \angle CDE$, where O is the circumcentre of ABC. Determine $\angle BAC$.

b. The pentagons $P_1P_2P_3P_4P_5$ and $I_1I_2I_3I_4I_5$ are cyclic, where I_i is the incentre of the triangle $P_{i-1}P_iP_{i+1}$ (reckoned cyclically, that is $P_0 = P_5$ and $P_6 = P_1$). Show that the lines P_1I_1, P_2I_2, P_3I_3, P_4I_4, and P_5I_5 meet in a single point.