The Niels Henrik Abel mathematics competition: Final 2024–2025

18 March 2025 (English)

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In the final round of the Abel contest there are four problems (eight subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

Allowed aids are writing paper, bilingual dictionaries, and writing tools including compass and ruler, but not protractor.

Problem 1

a. Peer and Solveig are playing a game with n coins, all of which show M on one side and K on the opposite side. The coins are laid out in a row on the table. Peer and Solveig alternate taking turns. On his turn, Peer may turn over one or more coins, so long as he does not turn over two adjacent coins. On her turn, Solveig picks precisely two adjacent coins and turns them over. When the game begins, all the coins are showing M. Peer plays first, and he wins if all the coins show K simultaneously at any time. Find all $n \geq 2$ for which Solveig can keep Peer from winning.

b. In Duckville there is a perpetual trophy with the words "Best child of Duckville" engraved on it. Each inhabitant of Duckville has a non-empty list (which never changes) of other inhabitants of Duckville. Whoever receives the trophy gets to keep it for one day, and then passes it on to someone on their list the next day. Gregers has previously received the trophy. It turns out that each time he does receive it, he is guaranteed to receive it again exactly 2025 days later (but perhaps earlier, as well). Hedvig received the trophy today. Determine all integers $n > 0$ for which we can be *absolutely certain* that she cannot receive the trophy again in n days, given the above information.

Problem 2

a. A teacher asks each of eleven pupils to write a positive integer with at most four digits, each on a separate yellow sticky note. Show that if all the numbers are different, the teacher can always select two or more of the eleven stickies so that the average of the numbers on the selected notes is not an integer.

b. Which positive integers a have the property that $n! - a$ is a perfect square for infinitely many positive integers n?

Problem 3

a. In the triangle ABC, the foot of the altitude from B is E, and the foot of the altitude from C is F. The lines orthogonal to the line EF through B and C intersect EF in P and Q, respectively. Show that $EP = FQ$.

b. An acute triangle ABC has circumcentre O. The lines AO and BC intersect in D, while BO and CA intersect in E, and CO and AB intersect in F. Show that, if the triangles ABC and DEF are similar (with vertices in that order), then ABC is equilateral.

Problem 4

a. Find all polynomials P with real coefficients satisfying

$$P\left(\frac{1}{1+x}\right) = \frac{1}{1+P(x)}$$

for all real numbers $x \neq -1$.

b. Determine the largest real number C such that

$$\frac{1}{x} + \frac{1}{2y} + \frac{1}{3z} \geq C$$

for all real numbers $x, y, z \neq 0$ satisfying the equation

$$\frac{x}{yz} + \frac{4y}{zx} + \frac{9z}{xy} = 24.$$