English

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Final round   6 March 2008

In the final round of the Abel contest there are 4 problems (8 subproblems) to be solved in 4 hours. You are required to show the reasoning behind your answers. Start a new sheet of paper for each problem.

The maximum score is 10 points for each problem. The total score is thus between 0 and 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

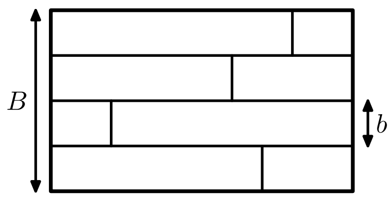
Problem 1

Let $s(n) = \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n$.

(a) Show that $s(n)$ is an integer whenever n is an integer.

(b) How many integers n with $0 < n \leq 2008$ are such that $s(n)$ is divisible by 4?

Problem 2



(a) We wish to lay down boards on a floor with width B in the direction across the boards. We have n boards of width b, and B/b is an integer, and $nb \leq B$. There are enough boards to cover the floor, but the boards may have different lengths. Show that we can cut the boards in such a way that every board length on the floor has at most one join where two boards meet end to end.

(b) A and B play a game on a square board consisting of $n \times n$ white tiles, where $n \geq 2$. A moves first, and the players alternate taking turns. A move consists of picking a square consisting of 2×2 or 3×3 white tiles and colouring all these tiles black. The first player who cannot find any such squares has lost. Show that A can always win the game if A plays the game right.

Problem 3

(a) Let x and y be positive numbers such that $x + y = 2$. Show that

$$\frac{1}{x} + \frac{1}{y} \leq \frac{1}{x^2} + \frac{1}{y^2}.$$

(b) Let x, y, and z be positive numbers such that $x + y + z = 2$. Show that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{9}{4} \leq \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}.$$

Problem 4

Note that the two subproblems (a) and (b) are unrelated, and the triangles in these subproblems do not need to be the same.

(a) Three distinct points A, B, and C lie on a circle with centre at O. The triangles AOB, BOC, and COA have equal area. What are the possible magnitudes of the angles of the triangle ABC?

(b) A point D lies on the side BC, and a point E on the side AC, of the triangle ABC, and BD and AE have the same length. The line through the centres of the circumscribed circles of the triangles ADC and BEC crosses AC in K and BC in L. Show that KC and LC have the same length.