English

The Niels Henrik Abel mathematics competition 2011–2012

Final 8 March 2012

In the final round of the Abel contest there are four problems (eight subproblems) to be solved in four hours. You are required to justify your answers. Start a new sheet of paper for each problem.

The maximum score is 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

a. Berit has 11 twenty kroner coins, 14 ten kroner coins, and 12 five kroner coins. An exchange machine can exchange three ten kroner coins into one twenty kroner coin and two five kroner coins, and the reverse. It can also exchange two twenty kroner coins into three ten kroner coins and two five kroner coins, and the reverse.

(i) Can Berit get the same number of twenty kroner and ten kroner coins, but no five kroner coins?

(ii) Can she get the same number each of twenty kroner, ten kroner, and five kroner coins?

b. Every integer is painted white or black, so that if m is white then $m+20$ is also white, and if k is black then $k+35$ is also black. For which n can exactly n of the numbers 1, 2, …, 50 be white?

Problem 2

a. Two circles S_1 and S_2 are placed so that they do not overlap each other, neither completely nor partially. They have centres in O_1 and O_2, respectively. Further, L_1 and M_1 are different points on S_1 so that O_2L_1 and O_2M_1 are tangent to S_1, and similarly L_2 and M_2 are different points on S_2 so that O_1L_2 and O_1M_2 are tangent to S_2. Show that there exists a unique circle which is tangent to the four line segments O_2L_1, O_2M_1, O_1L_2, and O_1M_2.

b. Four circles S_1, S_2, S_3 og S_4 are placed so that none of them overlap each other, neither completely nor partially. They have centres in O_1, O_2, O_3, and O_4, respectively. For each pair (S_i, S_j) of circles, with $1 \le i < j \le 4$, we find a circle S_{ij} as in part **a**. The circle S_{ij} has radius R_{ij}. Show that

$$\frac{1}{R_{12}} + \frac{1}{R_{23}} + \frac{1}{R_{34}} + \frac{1}{R_{14}} = 2\left(\frac{1}{R_{13}} + \frac{1}{R_{24}}\right).$$

Problem 3

a. Find the last three digits in the product $1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot 2009 \cdot 2011$.

b. Which positive integers m are such that $k^m - 1$ is divisible by 2^m for all odd numbers $k \geq 3$?

Problem 4

a. Two positive numbers x and y are given. Show that

$$\left(1 + \frac{x}{y}\right)^3 + \left(1 + \frac{y}{x}\right)^3 \geq 16.$$

b. Positive numbers $b_1, b_2, \ldots, b_n$ are given so that

$$b_1 + b_2 + \cdots + b_n \leq 10.$$

Further, $a_1 = b_1$ and $a_m = sa_{m-1} + b_m$ for $m > 1$, where $0 \leq s < 1$. Show that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq \frac{100}{1 - s^2}.$$