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English

The Niels Henrik Abel mathematics competition 2014–2015

Final 17 March 2015

In the final round of the Abel contest there are four problems (seven subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

a. Find all triples $(x, y, z) \in \mathbb{R}^3$ satisfying the equations

$$x^2 + 4y^2 = 4zx,$$
$$y^2 + 4z^2 = 4xy,$$
$$z^2 + 4x^2 = 4yz.$$

b. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f\big(yf(x)\big) = y^2 f(x) f\big(f(x)\big)$$

for all real numbers x and y.

Problem 2

a. King Arthur is placing $a + b + c$ knights around a table. a knights are dressed in red, b knights are dressed in brown, and c knights are dressed in orange.

Arthur wishes to arrange the knights so that no knight is seated next to someone dressed in the same colour as himself. Show that this is possible if, and only if, there exists a triangle whose sides have lengths $a + \frac{1}{2}$, $b + \frac{1}{2}$, and $c + \frac{1}{2}$.

b. Nils is playing a game with a bag originally containing n red and one black marble. He begins with a fortune equal to 1.

In each move he picks a real number x with $0 \le x \le y$, where his present fortune is y. Then he draws a marble from the bag. If the marble is red, his fortune increases by x, but if it is black, it decreases by x. The game is over after n moves when there is only a single marble left.

In each move Nils chooses x so that he ensures a final fortune greater or equal to Y. What is the largest possible value of Y?

Problem 3

The five sides of a regular pentagon are extended to lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, and ℓ_5. Denote by d_i the distance from a point P to ℓ_i. For which point(s) in the interior of the pentagon is the product $d_1 d_2 d_3 d_4 d_5$ maximal?

Problem 4

a. Determine all nonnegative integers x and y so that $3^x + 7^y$ is a perfect square and y is even.

b. Determine all nonnegative integers x and y so that $3^x + 7^y$ is a perfect square and y is odd.