The Niels Henrik Abel mathematics competition: Final 2018–2019

5 March 2019 (English)

In the final round of the Abel contest there are four problems (six subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

You have an $n \times n$ grid of empty squares. You place a cross in all the squares, one at a time. When you place a cross in an empty square, you receive $i + j$ points if there were i crosses in the same row and j crosses in the same column before you placed the new cross. Which are the possible total scores you can get?

Problem 2

Find all pairs (m, n) of natural numbers such that $mn - 1 \mid n^3 - 1$.

Problem 3

a. Three circles are pairwise tangent, with none of them lying inside another. The centres of the circles are the corners of a triangle with circumference 1. What is the smallest possible value for the sum of the areas of the circles?

b. Find all real functions f defined on the real numbers except zero, satisfying $f(2019) = 1$ and

$$f(x)f(y) + f\left(\frac{2019}{x}\right)f\left(\frac{2019}{y}\right) = 2f(xy)$$

for all $x, y \neq 0$.

Problem 4

The diagonals of a convex quadrilateral $ABCD$ intersect at E. The triangles ABE, BCE, CDE and DAE have centroids K, L, M and N, and orthocentres Q, R, S and T. Show that the quadrilaterals $QRST$ and $LMNK$ are similar.