Telenor

The Niels Henrik Abel Contest 1996–97
FINAL

Problem 1

A positive integer n is called happy if $n = a^2 + b^2$ for some integers a and b. Let t be a happy number.

a) Show that $2t$ is happy. (5 points)

b) Show that $3t$ is not happy. (5 points)

Problem 2

a) Let ABC be an equilateral triangle, and let P be a point in its interior. Let Q, R and S be the feet of the perpendiculars from P to AB, BC and CA respectively. Show that the sum $PQ + PR + PS$ is independent of the choice of P. (5 points)

b) Let A, B and C be three different points on a circle such that $AB = AC$. Let E be a point on the line segment BC, and let D be the point of intersection between the circle and the extension of AE ($D \neq A$). Show that the product $AE \cdot AD$ is independent of the choice of E. (5 points)

Problem 3

a) Each subset of 97 out of 1997 given real numbers has positive sum. Show that the sum of all the 1997 numbers is positive. (3 points)

b) In a school with 91 students distributed in 3 classes every student took part in a competition. For every sample of 6 students of the same sex at least 2 achieved the same number of points. Show that there are 4 students who are in the same class, are of the same sex, and who achieved the same result in the competition. (7 points)

Problem 4

Let $p(x)$ be a polynomial with integer coeffisients. Suppose there are different integers a and b such that $p(a) = b$ and $p(b) = a$. Show that the equation $p(x) = x$ has at most one integer solution. (10 points)