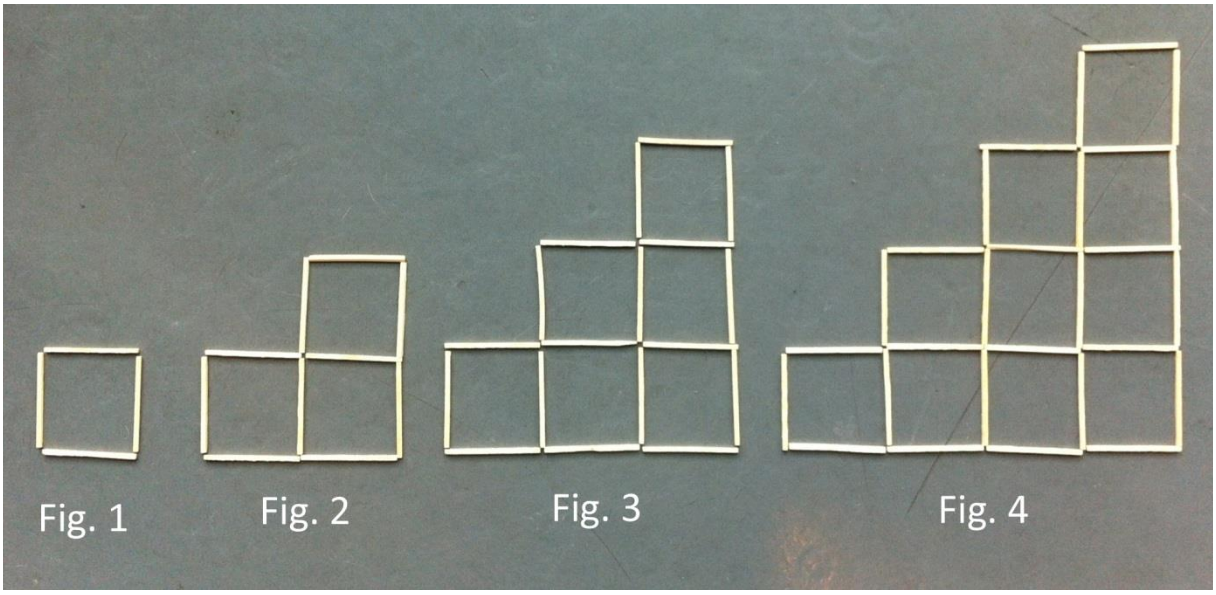
Staircase of Sticks

The picture shows a figure that can be expanded an infinite number of times.



A Subject Report

You should investigate

a) the perimeter of the figures, measured in number of sticks
b) the number of sticks required for each figure

Find a relation between the figure number and both the number of sticks needed for the perimeter (a) and the number of sticks needed for the whole figure (b).

1. Solve the problem in as many ways as you can think of.
2. Make a joint subject report that carefully explains how you reached your various results.
3. Compare and contrast the different solutions. What is the same? What is different?
4. Which method do you think is the most efficient/effective? Give a reason for your answer.

B Exhibition and presentation

a) Make your own task using either two- or three-dimensional figures that grow according to a given pattern.

b) Find a pattern in nature, art, clothing design, architecture or similar and explain how the pattern is constructed.

C Progress log

The progress log should describe how the class has worked through tasks A and B.

Täby friskola log

Jan

28

We’re eagerly waiting for the results. Will we advance in Sigma? It’s silent. “WE’RE IN THE SEMIFINAL!” she yells.

We get the first task and get going. On the worksheet there’s a growing number of matches. They form the same pattern as a stair. We study it in pairs. We’re looking for equations explaining how the perimeter and the number of matches grow.

Feb

2

The work with the stairs goes on. Since we already have knowledge about correlations and triangular numbers it isn’t hard to find the formulas for the circumference and number of matches. The pairs get a paper where we jot down solutions and explanations. The pairs swap papers with each other and on notes write down good things and things that need improvements. We work with the improvements and clarifications of the text and tables.

5

We’re done with the stair and go on with creating our own growing patterns. We want to work tri-dimensionally because we seldom do that. To make it easier we build figures of blocks. We work in groups of four.

6

We study the figures that we’ve created and want to find one that we can work more with. It’s difficult to decide. Is it interesting enough? Instead of voting we decide to deepen our knowledge about the patterns that seems the most fun.

9

The last task is to find patterns in nature, architecture etc. Everyone comes up with suggestions that we discuss in groups. The groups hand in one suggestion for the project. Patterns for crystals, fractals and cobwebs are written down. Anna talks about them so that we can form an opinion. Fractals are the most popular, since it’s a modern pattern that we encounter in everyday life.

11

The work from 6/2 is resumed. Every group gets one of the figures that we’ve chosen. We have to find different ways of solving them. We find equations for all the patterns but one, the hardest: “The sum of squares”. What’s hard with this pattern is to find the formula that describes how it grows. One group comes with a last minute contribution. The lesson is over and the pattern is left unsolved.

12

Anna won’t give us the equation for ”The sum of the squares”, but shows us a picture that describes the Babylonians’ thoughts about the pattern. It’s hard to understand the picture but we begin to see correlations when Anna asks us to describe it. Two pupils jump up from their seats. “We’ve found the formula!”

13

Anna is about to show us the equation formula that will help us understand the pattern “The sum of the squares”. Two students explain how they interpreted the picture and the formula. It’s, once again, hard to understand, but Anna explains. 11/2 we found another pattern (the last minute contribution), that needs to be solved. It’s difficult but fun.

We begin and soon one of us comes up with a complicated equation formula. It needs simplifying and some try to find the equation on Google. They search for numbers that are found in the pattern. A link to Wikipedia about octahedral numbers appears. There it is! Our pattern but with another equation formula. Is our formula correct? We do some calculations and realize that it’s the same formula written differently. The lesson comes to an end and we vote on which pattern to go for. It’s a tie between “The sum of the squares” and “The square pyramid” (the octahedral number), but with seconds left we decide to work with “The square pyramid”. We’re content about the choice since we don’t know much about it and we want to learn more.

16

We work in groups with fractals, our nature pattern and we find a lot of new facts about them, e.g. who discovered them. Some of us are really absorbed with the pictures and videos that we find.

Mar

9

The presentations about fractals are due today. We think that the pictures of fractals are cool. There are many directions within the fractal field. We choose landscapes since it includes both real and fictional landscapes in games/films.

16

Today the mathematician Johan Thorbiörnson from KTH (Royal Institute of Technology) visits us. He is to talk about fractals but he also has something else planned. We already know some about the pattern he shows us, the Euler characteristic. Earlier we’ve talked about

Platonic solids and the correlation between vertices, edges and faces that Euler studied. The correlation is found everywhere. You can e.g. prove that Earth is a sphere. With 20 minutes left, we remind Johan that we want to hear about fractals. He tells us that fractals have a dimension between one and two. It’s interesting and we learn a lot of new things.

18

We have to make a choice. Fractals or the Euler characteristic? What Johan told us is fascinating and we discuss what to do. Since it’s easier to understand and explain we choose the Euler characteristic.

27

When we worked with Sigma it wasn’t always easy, it was hard to agree with strong wills involved. We’ve solved many difficult tasks together which has made us a tighter group. It’s been worthwhile and interesting despite all the hard work. We’ve had great fun!

THE STAIR OF MATCHES

NMCC SIGMA8 2015
TÄBY FRISKOLA 8 SPETS
SWEDEN

Content

Introduction ... **2**

Circumference ... **3**

Group 1 ... 3

Group 2 ... 4

Group 3 ... 4

Group 4 ... 5

Comparison of the methods - circumference ... **7**

The total number of matches ... **8**

Group 5 ... 8

Group 6 ... 11

Group 7 ... 12

Comparison of the methods – the total number of matches ... **14**

Comparison of the two relationships ... **15**

Conclusions ... **16**

References ... **17**

Introduction

Our task is to find a relationship for how the stair of matches shown below grows. We investigate how many matches are needed for a specific circumference and how many matches are needed in total for any figure.

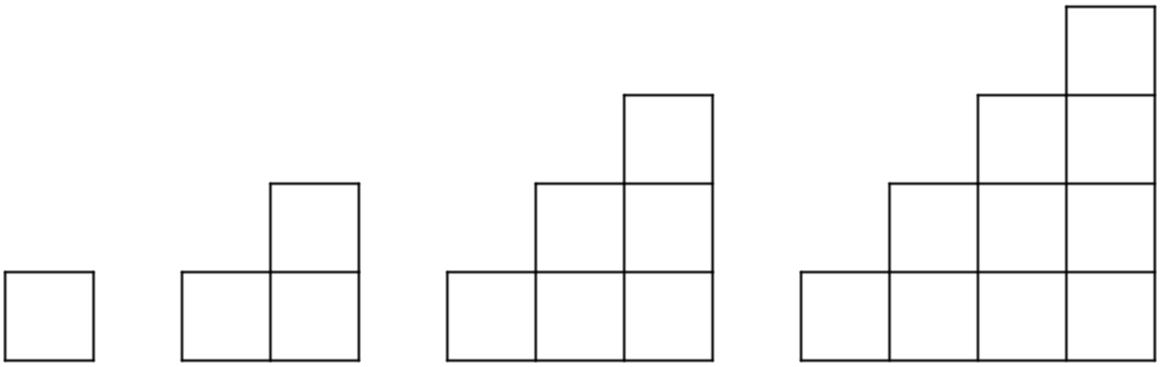


fig. 1 fig. 2 fig. 3 fig. 4

To be able to work with finding these relationships we split into smaller groups, working independently. This means that we get a wide set of solutions using different methods so that we can compare the pros and cons of different approaches.

Circumference

Group 1

Group 1 produces a table, and notes that the number of matches in the circumference is divisible by 4.

Number of the figure	Number of matches in the circumference
1	4
2	8
3	12
4	16
5	20
6	24
7	28

Table 1

The circumference increases with four matches for each figure, and the number of matches in the circumference is four times the number of the figure. They conclude that if

x is the number of the figure and

y is the number of matches in the circumference in any figure then

$y = 4x$

Group 2

Group 2 draws the same conclusion, but by a different reasoning. They derive the relationship between the circumference and figure by dividing the matches in the circumference in four groups that have different colours. In each group there are as many matches as the number of the figure. Thus the circumference is 4 times the number of the figure.

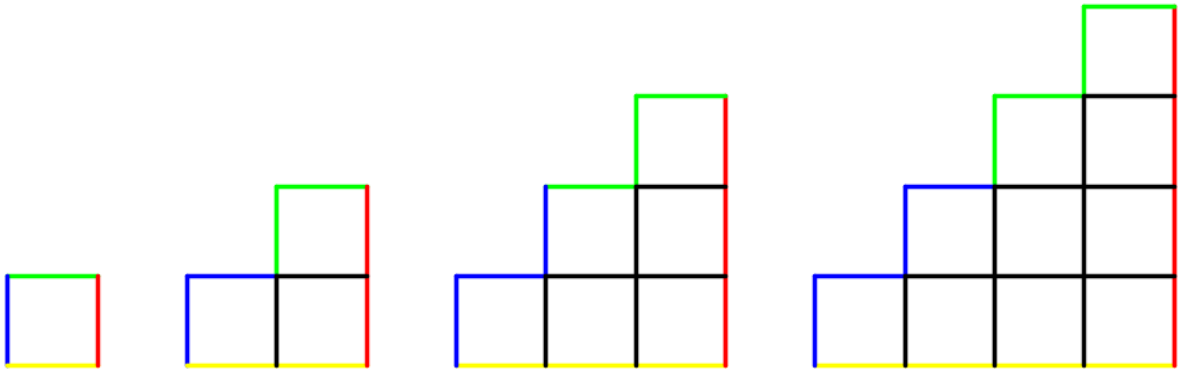


fig. 1 fig. 2 fig. 3 fig. 4

Picture 1

Group 3

Group 3 unfold the figures, e.g. figure 3. In this way it is easier to see the relationship (note that this transformation is only done for the matches in the circumference). When the circumference is transformed, they conclude that the matches form a square with each side having the number of matches equal to the figure number. This leads to the same formula as the previous groups have derived.

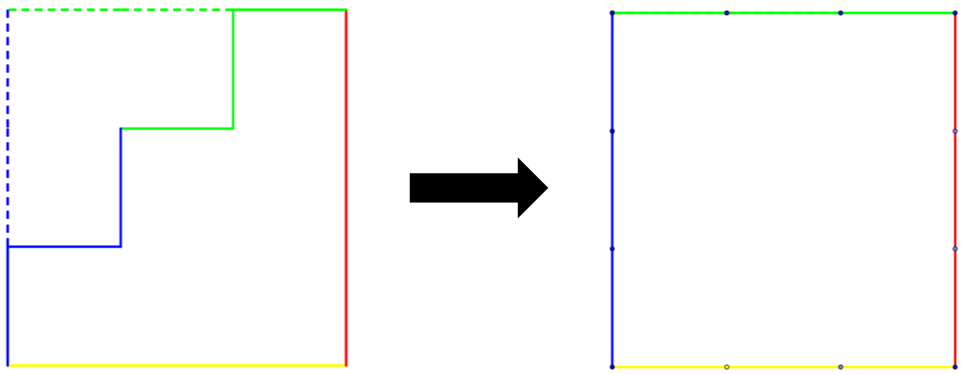


Fig. 3

Picture 1

Group 4

Group 4 starts by counting the number of matches in the circumference and arrange the result in a table (*table 2*).

Number of the figure (x)	Number of matches in the circumference (y)
1	4
2	8
3	12
4	16
5	20
6	24
7	28

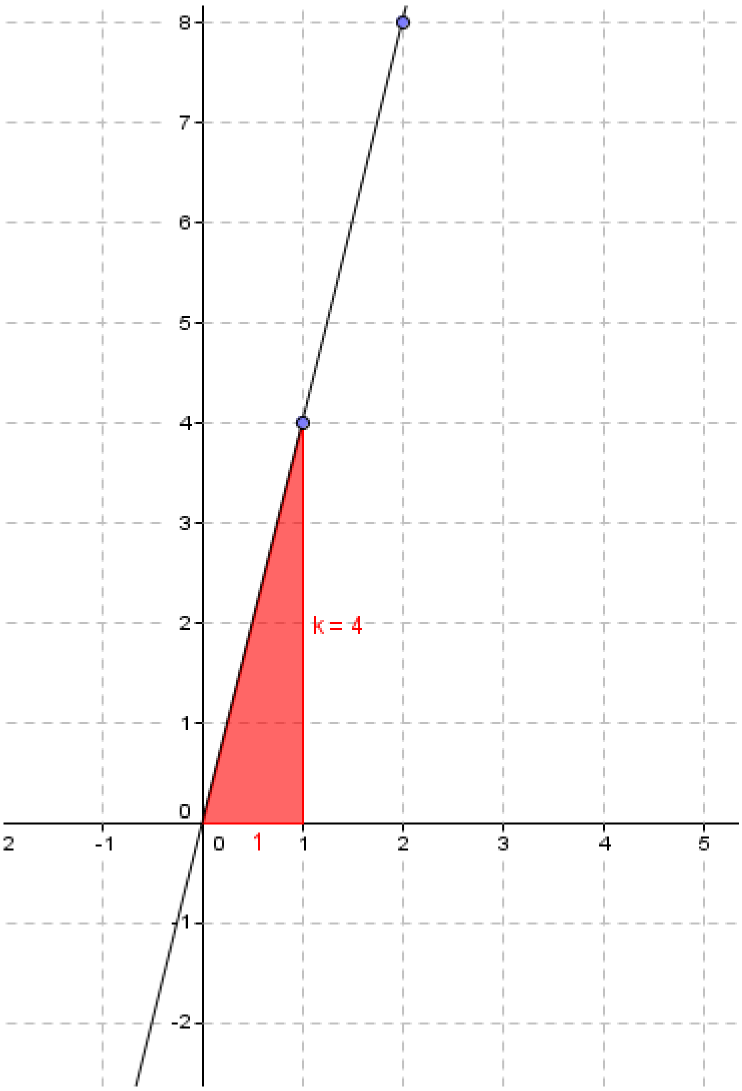
Table 2

Each row in the table corresponds to a point in a coordinate system. They realise that the points are on a line, a linear relationship. They draw a line through all points (*picture 3*). The general formula for a linear relationship is

$y = kx + m,$

where k is the slope of the line and m is the y-coordinate for the interception of the y-axis.

The line goes through the origin, the point with the coordinates (0,0). This means that m is zero. To calculate the slope, the group draws the red triangle. When x increases by one, y increases four times. So they conclude that k equals four.



Picture 3

They conclude that $y = 4x$.

Comparison of the methods – circumference

All groups start by investigating the pattern by counting the number of matches. Then the groups follow different directions. Some focus on the number of matches ordered in tables, other use figures.

Group 1 uses a table (*table 1*) to easily discover the circumference relationship.

Group 2 divides the circumference into four groups, where each group has as many matches as the number of the figure (*picture 1*).

Group 3 also divides the matches into groups, but after having transformed the circumference to a square (*picture 2*).

Group 4 creates a graph (*picture* 3) and notes a linear relationship. They use the equation for a linear relationship to calculate the formula.

We have solved the problem and now we want to compare the solutions.

We start by examining the solutions where the matches have been divided into groups. In our opinion, the solution where the figure is transformed into a square is the easiest to follow, since the square shows a simple grouping of the matches. In the picture created by Group 2 it is more difficult to see the grouping since the picture is more confusing. The solution where the figure is transformed works in this case, but it is unlikely that any pattern can be transformed in this way to a regular figure. We were just lucky that the pattern could be transformed. The method used by Group 2 is more general and can be applied to other problems.

The most efficient method is the one used by Group 1. Since the problem is not that complex, it is sufficient to look at the numbers in the table to see the relationship between the number of the figure and the circumference. This solution works on simple problems.

The method used by Group 4 is efficient since they only need to count the number of matches and mark the points in a coordinate system. This requires more knowledge about linear equations to derive the formula from the graph.

The total number of matches

Group 5

Group 5 focuses on groups in the figures. They create three groups in each figure. Two of the groups consist of the triangular number corresponding to the figure number.

Triangular numbers

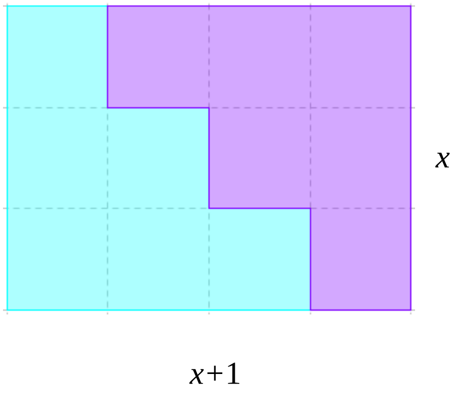
The triangular number is calculated by adding the figure number of all previous figure numbers, for example the fifth triangular number is calculated by adding 1,2,3,4 and 5. The sum is 15, the fifth triangular number.

Figure number	Calculation	Triangular number
1	1	1
2	1+2	3
3	1+2+3	6
4	1+2+3+4	10
5	1+2+3+4+5	15
6	1+2+3+4+5+6	21
7	1+2+3+4+5+6+7	28
8	1+2+3+4+5+6+7+8	36
9	1+2+3+4+5+6+7+8+9	45
10	1+2+3+4+5+6+7+8+9+10	55

Table 3

This method is time consuming, especially for large triangular numbers.

A formula for triangular numbers can be derived in different ways. One way is to construct a rectangle (*picture 4*). The shorter side has the length *x*, the length of longer side is equal to *x*+1. The triangular number of *x* is half of the rectangle's area, so we multiply the sides and divide by two.



Picture 4

x is the number of the figure

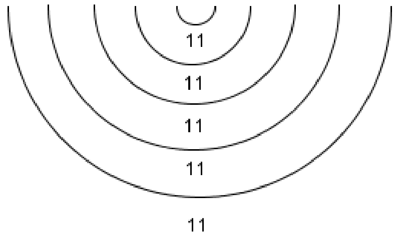
y is the triangular number

$$y = \frac{x(x+1)}{2}$$

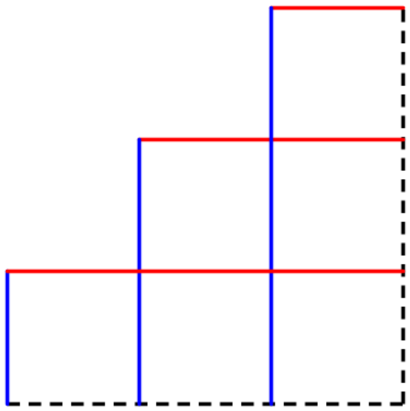
The formula can also be derived by a method created by Gauss. Gauss was an 18th century mathematician. His method uses the arithmetic sum. The first number is added to the last (the figure number), the next number is added to the second last number and so on.

We show this with 10 as the figure number

1+2+3+4+5+6+7+8+9+10 = 5*11 = 55 = 10*(10+1)/2



All additions result in 11(10 +1, 9+2,…, 6+5), the number of additions are 5, that is 10/2. When we multiply 11 with 5 we get the 10th triangular number, 55. This leads to the same formula that Group 5 uses.



Picture 5

The number of blue matches in, for example, the third figure, equals the third triangular number (6) and the same is true for the red matches. The third group, the dotted matches, is always twice as big as the figure number, in this case $3 \cdot 2 = 6$. The total number of matches is the triangular number of the figure multiplied by 2, plus the double figure number.

y is the number of matches

x is the number of the figure

$$y = 2 \cdot \frac{x(x+1)}{2} + 2x = x(x+1) + 2x = x^2 + x + 2x = x^2 + 3x$$

Group 6

Group 6 puts the number of the figure and the number of matches in a table. They choose to write the number of matches as a multiplication with the number of the figure as a factor. The other factor is always the number of the figure plus three.

Number of the figure	Factors	Number of matches
1	$1 \cdot 4 = 1 \cdot (1+3)$	4
2	$2 \cdot 5 = 2 \cdot (2+3)$	10
3	$3 \cdot 6 = 3 \cdot (3+3)$	18
4	$4 \cdot 7 = 4 \cdot (4+3)$	28
5	$5 \cdot 8 = 5 \cdot (5+3)$	40
6	$6 \cdot 9 = 6 \cdot (6+3)$	54

Table 4

The formula is now easy to discover!

y is the number of matches

x is the number of the figure

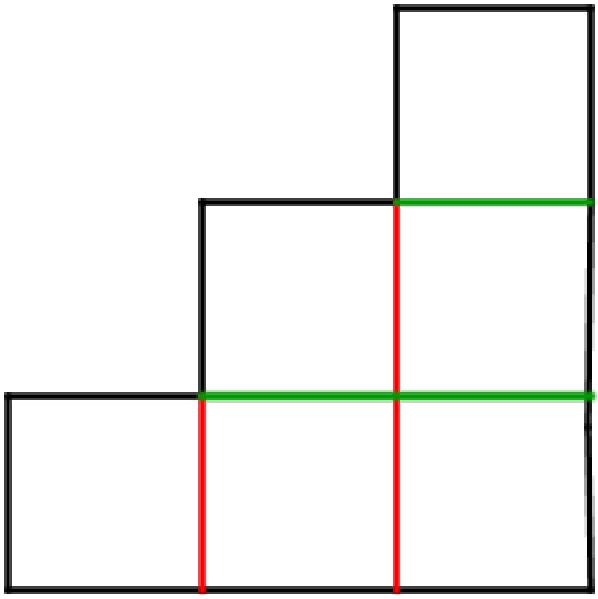
Formula:

$$y = x(x+3) = x^2 + 3x$$

Group 7

Group 7 starts with the formula for the circumference. Then they try to calculate the number of matches remaining in any figure. Each group has the number of matches of the figure number and there is always one group less than the figure number.

Fig. 3

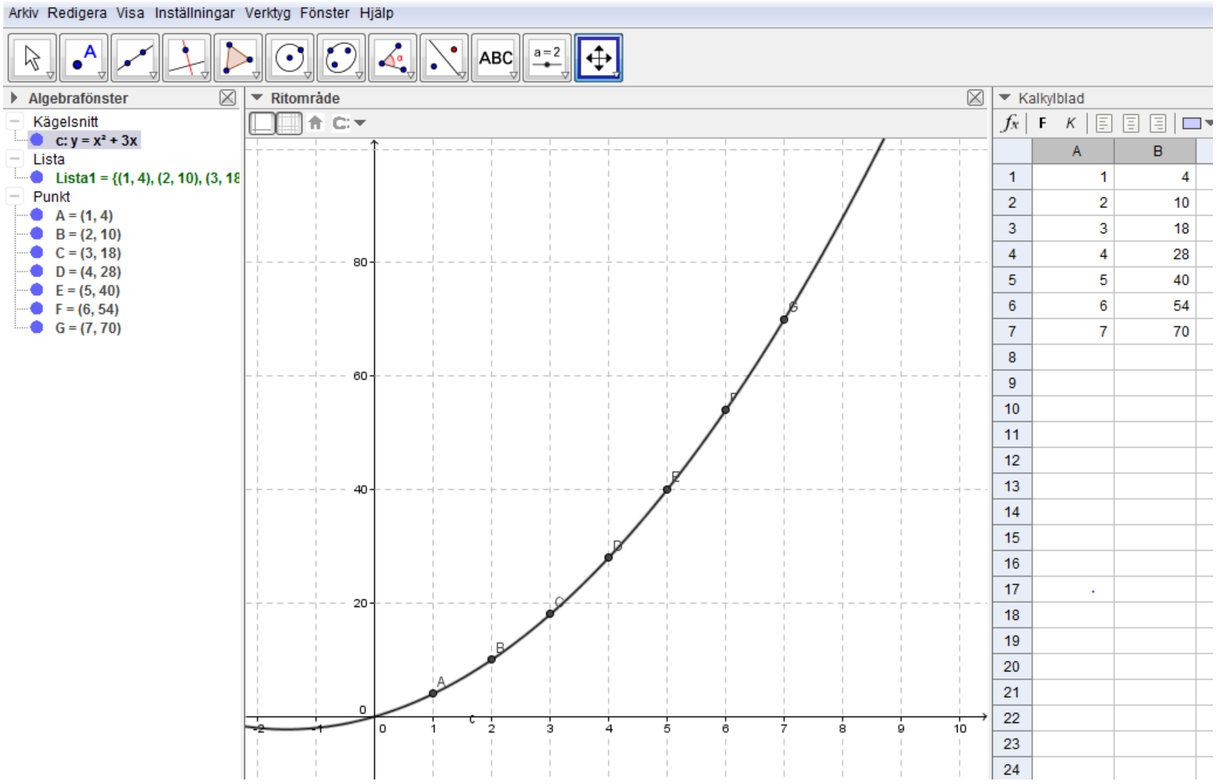


Picture 6

The formula is:

$y = 4x + x(x - 1) = 4x + x^2 - x = x^2 + 3x$

Group 7 uses the software GeoGebra to plot a graph. They input a table where one column is the figure number; the other is the number of matches. The table is used to mark points in the coordinate system. They want to confirm that their formula is correct and inputs the formula as well. The graph goes through all the points of the table. This shows that the formula is correct. If not all points had been on the graph something must be wrong with the formula.


Arkiv Redigera Visa Inställningar Verktyg Fönster Hjälp
Algebrafönster
Kägelsnitt
c: y = x² + 3x
Lista
Lista1 = {(1, 4), (2, 10), (3, 18
Punkt
A = (1, 4)
B = (2, 10)
C = (3, 18)
D = (4, 28)
E = (5, 40)
F = (6, 54)
G = (7, 70)
Ritområde
Kalkylblad
A B
1 1 4
2 2 10
3 3 18
4 4 28
5 5 40
6 6 54
7 7 70

Picture 7

Comparison of the methods – number of matches

Group 5 uses triangular numbers to derive a formula for the total number of matches in a figure (*picture 5*).

Group 6 puts the number of matches in a table and by inspection finds a factorization of the number of matches (*table 4*).

Group 7 uses the formula for the circumference and then calculates the number of remaining matches by grouping the matches (*picture 6 and 7*).

Of all the three methods we think that the most efficient is the one used by Group 5. They use triangular numbers. This solution is the easiest from our point of view, since we already had knowledge about triangular numbers. We also find it good since the grouping in each figure is independent of the size of the figure. If you do not have knowledge about triangular numbers it is probably hard to derive a formula in this way.

The method used by Group 6 is the least demanding, the only required knowledge is multiplication.

Group 7 uses the most time demanding method. To use the circumference and then another formula is less efficient since you need to think twice. Also, the grouping used required some trial and error. Since they felt unsure about the formula they used GeoGebra to verify the result.

Comparison of the two relationships

The graphs of the two relationships are different, one (the graph for the circumference) is a straight line, and the other is not (the graph for the total number of matches). The explanation can be found in table 5 and 6.

The difference between the number of matches in the circumference is constant. The graph is a straight line since the slope never changes. The number of matches in the circumference is proportional to the figure number.

Number of the figure	Number of matches in the circumference	The difference between the number of matches in the circumference
1	4	
2	8	4
3	12	4
4	16	4
5	20	4

Table 5

The difference of the total number of matches increases with the number of the figure. The difference of the difference is however constant. We realise that the points cannot be on a straight line, the relationship is not linear.

Number of the figure	Number of matches	The difference between the number of matches	The difference of the difference
1	4		
2	10	6	
3	18	8	2
4	28	10	2
5	40	12	2

Table 6

Conclusions

We have studied the stair of matches. By investigating the circumference and the total number of matches in any figure, we have derived relationships and stated formulas. Initially we divided the class into smaller groups that worked independently. In this way we got several different solutions. We compared the solutions and discussed pros and cons of the different approaches. The different methods vary in how easy they are to understand and use, depending on our pre-knowledge. We cannot really say which method is the best, it depends on the person who uses it.

In conclusion, we think that this was a very interesting problem, since there are so many ways to solve it and therefore develop our mathematical knowledge. We had a lot of interesting discussions, we all learned many things and gained new experiences.

References

Books:

- Diskret matematik -*Matematik 3000 (2005)* Bokförlaget natur och kultur
- Bengt Ulin -*Liten guide för matematiska problemlösare (1993)* Liber
- Bengt Ulin -*Matematiska äventyr (2010)* NCM
- Jan Unenge -*Människorna bakom matematiken (1997)* Studentlitteratur

Websites:

- http://sv.wikipedia.org/wiki/Proportionalitet_%28matematik%29
- http://sv.wikipedia.org/wiki/Carl_Friedrich_Gauss

People:

- Erland Arctaedius
- Johan Thorbiörnson

Photo:

- Linda Nyholm