NMCC Sigma 8

ASSIGNMENT

REPORT

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Sweden

2016

Introduction

The assignment for Sigma8 2016 is to examine a growing pattern. We also answer several questions about the pattern. To solve the assignment we create several different solutions using different methods, which we then compare to each other. This way we see different relationships between the methods, and are able to draw joint conclusions.

We work with the assignment in independent pairs. We begin by studying the relationship between the figure´s number and the total number of small cubes. Together we produce 19 solutions, using nine different methods. We select five of these methods to present in our report, as they are the ones best suited for this task. The illustration below shows the first three figures of the pattern we work with.

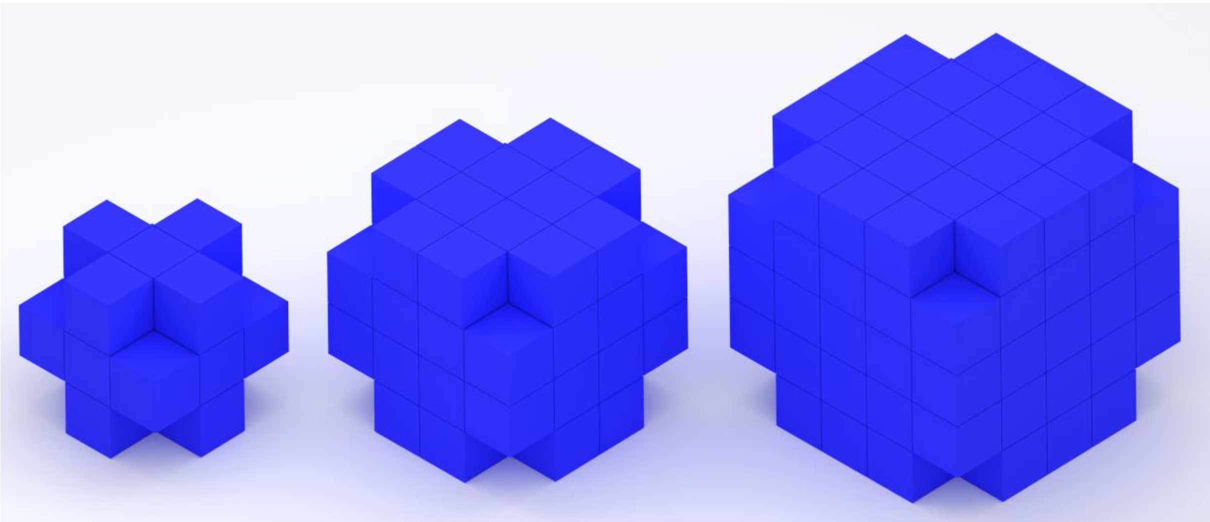


Image 1

Next, we examine the relationship between the figure's number and the number of small cubes with zero, one, two, three or four visible sides. To find the relationships we are looking for, we once again work in pairs. We come up with several interesting solutions, presented on the following pages.

The relation between the figure's number and the number of small cubes

We use the following variables throughout the report:

x : the figure's number

y : the total number of small cubes

Method 1

We consider the figure as a large cube whose small corner cubes are removed. In order to facilitate the calculation we include the eight non-existent corner cubes to get a complete cube. The number of small cubes along this cube's edge is always two more than the figure's number, $(x + 2)$ and the cube's volume is $(x + 2)(x + 2)(x + 2)$, which is the same as $(x + 2)^3$. Given that a cube always has eight vertices, we subtract the term with eight to compensate for the missing corner cubes.

The formula for the entire figure:

$$y = (x + 2)^3 - 8$$

$$= (x^2 + 4x + 4)(x + 2) - 8$$

$$= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8 - 8$$

$$= x^3 + 6x^2 + 12x$$

Method 2

We peel off the two opposite outer parts of the figure, orange in *Image 2* and left is a cuboid. The cuboid's edges are $(x + 2)$, $(x + 2)$ and x. The volume is $x\,(x + 2)^2$. Each peeled off part consists of a blue cuboid and two orange rods, see *Image 3*. The edges of the blue cuboid are 1, x and $(x + 2)$. Each rod is made up of x small cubes. The volume for both the peeled off pieces is $2\,(1 \cdot x\,(x + 2) + 2\,x)$.

The formula for the entire figure:

$y = x(x + 2)^2 + 2(1 \cdot x\,(x + 2) + 2x)$

$= x(x^2 + 4x + 4) + 2(2x + x^2 + 2x)$

$= x^3 + 4x^2 + 4x + 4x + 2x^2 + 4x$

$= x^3 + 6x^2 + 12x$

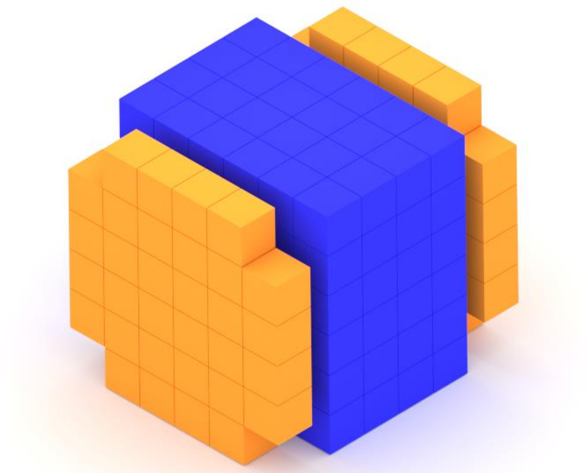


Image 2

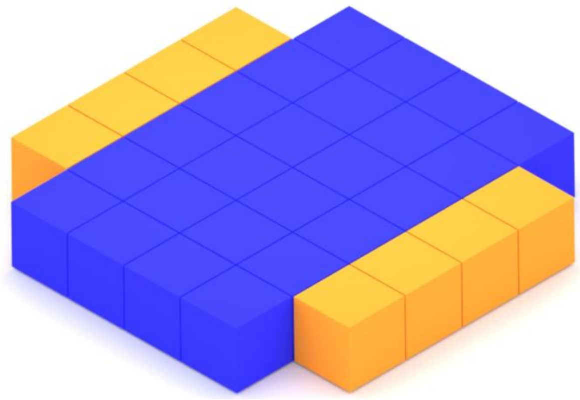


Image 3

Method 3

We divide the image in $(x + 2)$ slices as *Image 4* shows, so that x slices are blue congruent square cuboid with edges $(x + 2)$, $(x + 2)$ and 1. The total number of small cubes in this cuboid is $x (x + 2)^2$.

The remaining two slices are congruent and consists of a blue square cuboid and four orange rods that are placed on each side of the square, see *Image 5*. The edges of the blue squared formed cuboids are x, x, and 1. Therefore the volume is $x \cdot x \cdot 1 = x^2$ small cubes. It remains to calculate the four rods. Each rod is made of x small cubes. Expression of the two slices is $2 (x^2 + 4x)$.

The formula for the entire figure:

$y = x(x + 2)^2 + 2(x^2 + 4x)$

$= x(x^2 + 4x + 4) + 2(x^2 + 4x)$

$= x^3 + 4x^2 + 4x + 2(x^2 + 4x)$

$= x^3 + 4x^2 + 4x + 2x^2 + 8x$

$= x^3 + 6x^2 + 12x$

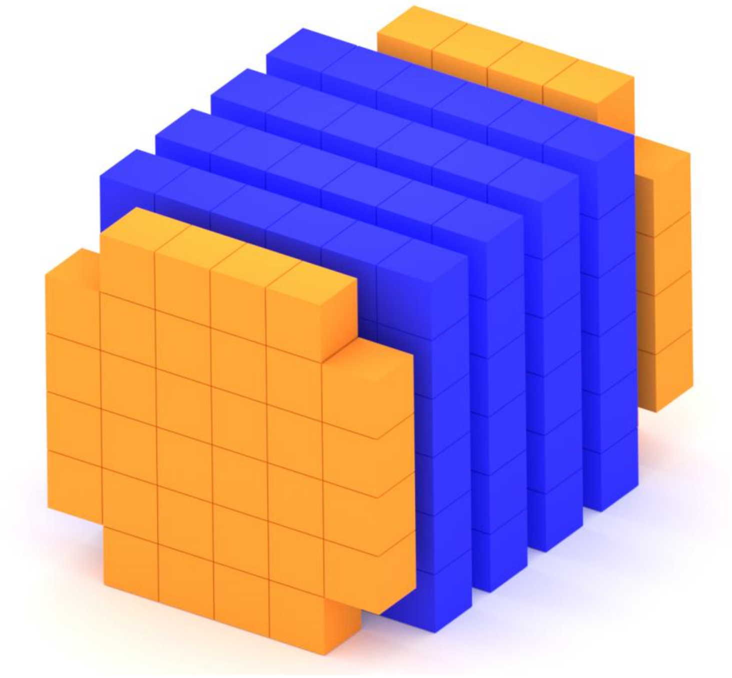


Image 4

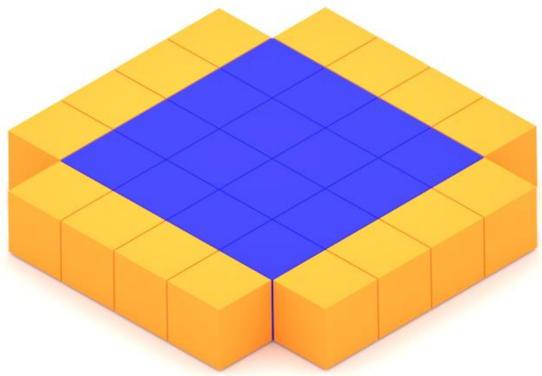


Image 5

Method 4

In this method we use triangular numbers.

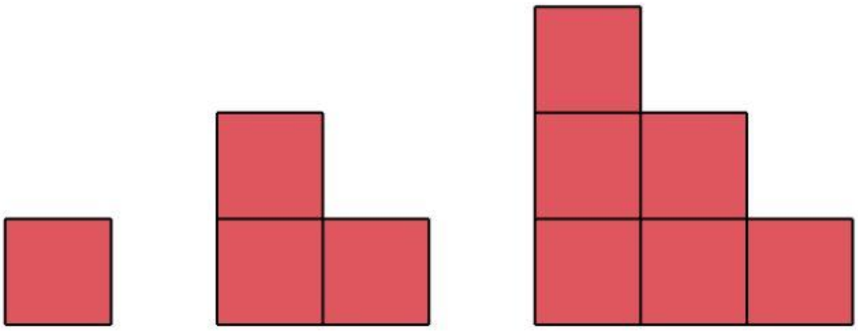
Triangular number is a pattern that grows by the figure's number which is added to the previous triangular number, see *Table 1*.

n	Calculation	Triangular Number
1	1	1
2	1+2	3
3	1+2+3	6
4	1+2+3+4	10
5	1+2+3+4+5	15

Table 1

n: Number of Triangular Number

We create an image of the pattern showing the first three Triangular Numbers (*Image 6*).

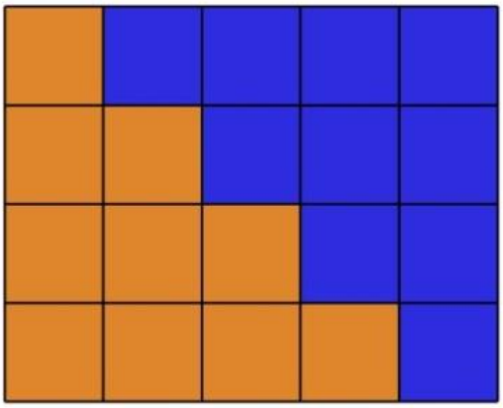


n = 1 *n = 2* *n = 3*

Image 6

Two equal triangular numbers can create a rectangle, as *Image 7* shows.

(n + 1)



n

Image 7

The height of the rectangle is equal to the number of the triangular number (n). The base of this rectangle is always one unit of length longer than the height ($n + 1$) and the rectangle's area is $n (n + 1)$. The triangular number is half the size of the rectangle, which gives us the expression for any triangular number, $\frac{n(n+1)}{2}$.

We divide the pattern figure in $(x + 8)$ pieces, i.e. twelve pieces for figure number 4 in the pattern sequence, see *Image 8*.

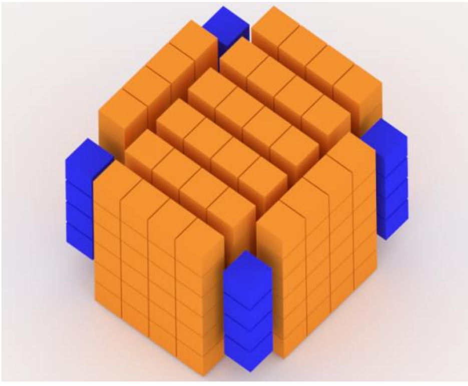


Image 8

Four of these pieces are blue rods. Each rod is made up of x small cubes. The other pieces are congruent orange cuboids with a width of a small cube. The number of such cuboid is four more than the figure's number, $(x + 4)$.

Each orange cuboid can be divided into three, see *Image 9*. In the purple group there is always x small cubes. The other two groups are the same triangular numbers with image number x. The total number of small cubes in an orange cuboid is

$2 \cdot \frac{x(x+1)}{2} + x = x(x + 1) + x.$

Expression of all orange cuboids together is $(x (x + 1) + x) (x + 4)$.

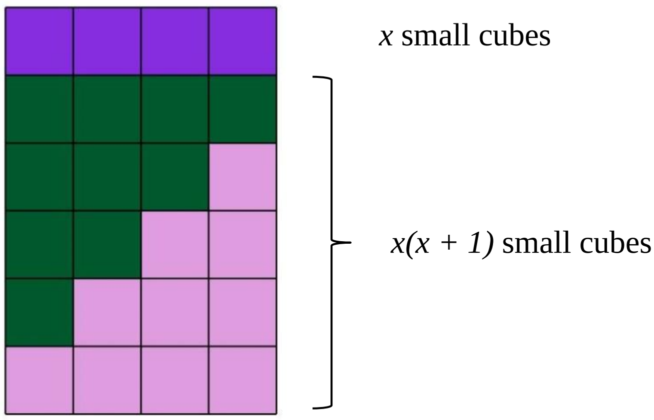


Image 9

The formula for the entire figure:

$y = (x (x + 1) + x)(x + 4) + 4x$

$=(x^2 + 2x)(x + 4) + 4x$

$= x^3 + 4x^2 + 2x^2 + 8x + 4x$

$= x^3 + 6x^2 + 12x$

Method 5

By using *Table 2*, we see the difference between the numbers in Column B, for the number of small cubes. For each new column, we see the difference between the numbers in the previous columns. This is done until the difference has become constant, which in this case is at the third difference. For this reason, we know that the biggest exponent is three.

A	B	C	D	E
Figure number	**Number of small cubes**	**First difference**	**Second difference**	**Third difference**
1	19			
2	56	37		
3	117	61	24	
4	208	91	30	6
5	335	127	36	6
6	504	169	42	6
7	721	217	48	6

Table 2

Since we now know we have the exponent three in the formula, we relate it to a cubic number, x^3. We associate with a cubic numbers, because they're the easiest expressions with exponent 3.

In *Table 3*, we compare cubic numbers, C*olumn B*, with the number of small cubes in the image, *Column D*, but find no clear connection. This means that the formula contains more than just x^3.

A	B	C	D	E	F
Figure number	**Cubic number**	**Diff between cubic numbers**	**Number of small cubes**	**Diff between number of small cubes**	$(x + 2)^3$
1	1		19		27
2	8	7	56	37	64
3	27	19	117	61	125
4	64	37	208	91	216
5	125	61	335	127	343
6	216	91	504	159	512
7	343	127	721	217	729

Table 3

Instead we compare *Column C,* which shows the difference between the cubic numbers, with *Column E,* which shows the difference between the number of small cubes, and we get a more interesting result. The columns have the same numbers in the same order, but the numbers in *Column C* are moved two steps down in the table. This means the difference between the number of small cubes in the entire image represents a figure number two greater than the difference between cubic number. This means the formula contains $(x + 2)^3$, but it may not be comprehensive. Therefore, we compare the expression $(x + 2)^3$, in *Column F*, with the number of small cubes in the figure, in *Column D*. We conclude that the numbers in *Column F* are eight higher than the numbers in *Column D,* so the formula is $(x + 2)^3 - 8$.

The formula for the entire figure:

$y = (x + 2)^3 – 8$

$= (x^2 + 4x + 4) (x + 2) - 8$

$= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8 - 8$

$= x^3 + 6x^2 + 12x$

Analysis

Our group believes *Method 1* is the best solution. It is the smartest, most common and most effective method. We think it is obvious the figure like a cube without corner cubes and then find a formula for the pattern. If the formula needs to be simplified M*ethod 1* is more demanding since few in our group master cubic expressions.

Method 2 is also an effective and smart solution, but does not reach the same level as *Method 1*. We created the solution by physically building the figure out of small cubes, since it is then easier to see the different parts. When we simplify the formula we do not encounter the same problems as in *Method 1*.

Method 3 is not advanced, because splitting the figure is easy to understand. Simplification of the expression in this method requires more focus because there are many steps to keep track of. The method is similar to *Method 2* and *4* in this respect that we split the figure into smaller parts. The splitting in *Method 4* is more complex, as there are more parts. *Method 4* is also more complicated as we need to know the formula for triangular numbers. In this aspect, *Method 4* is not the smartest solution according to us.

Method 5 requires most mathematical knowledge according to us, and is more advanced than all the previous methods. We believe more abstract thinking is required to analyze the table.

We are working with three different types of solutions; one type of solution in which we imagine the image as a simple geometric figure (*Method 1*), another as we split the figure into smaller parts (*Method 2*, *3* and *4*), and a third with the help of tables (*Method 5*). The first two types of solutions are more concrete in the sense that we can see and feel the figures and parts. These types of solutions, which are geometric, are good to use if we want to explain how we develop our expressions. A major advantage of our third type of solution is that it also works with only the number series, compared to the other solution types which require images.

The relationship between the figures number and number of visible sides

a: the number of small cubes with zero visible sides

b: the number of small cubes with one visible side

c: the number of small cubes with two visible sides

d: the number of small cubes with three visible sides

e: the number of small cubes with four visible sides

Zero visible sides

The small cubes with zero visible sides are the cubes inside the figure, *Image 10*. The small cubes are always in the form of a larger cube. The entire figure has the side length $(x + 2)$. When we peel off the sides to find the small cubes which have zero visible sides, one small cube from each side disappears. That is why the edge of the blue cube is x. We calculate the volume of a cube by multiplying the width, height and depth.

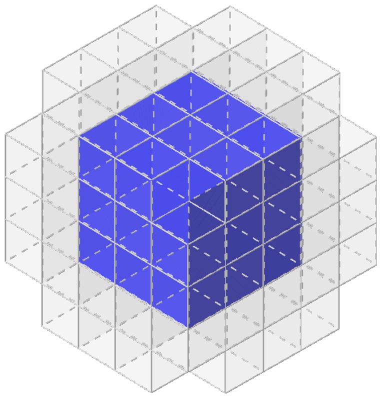
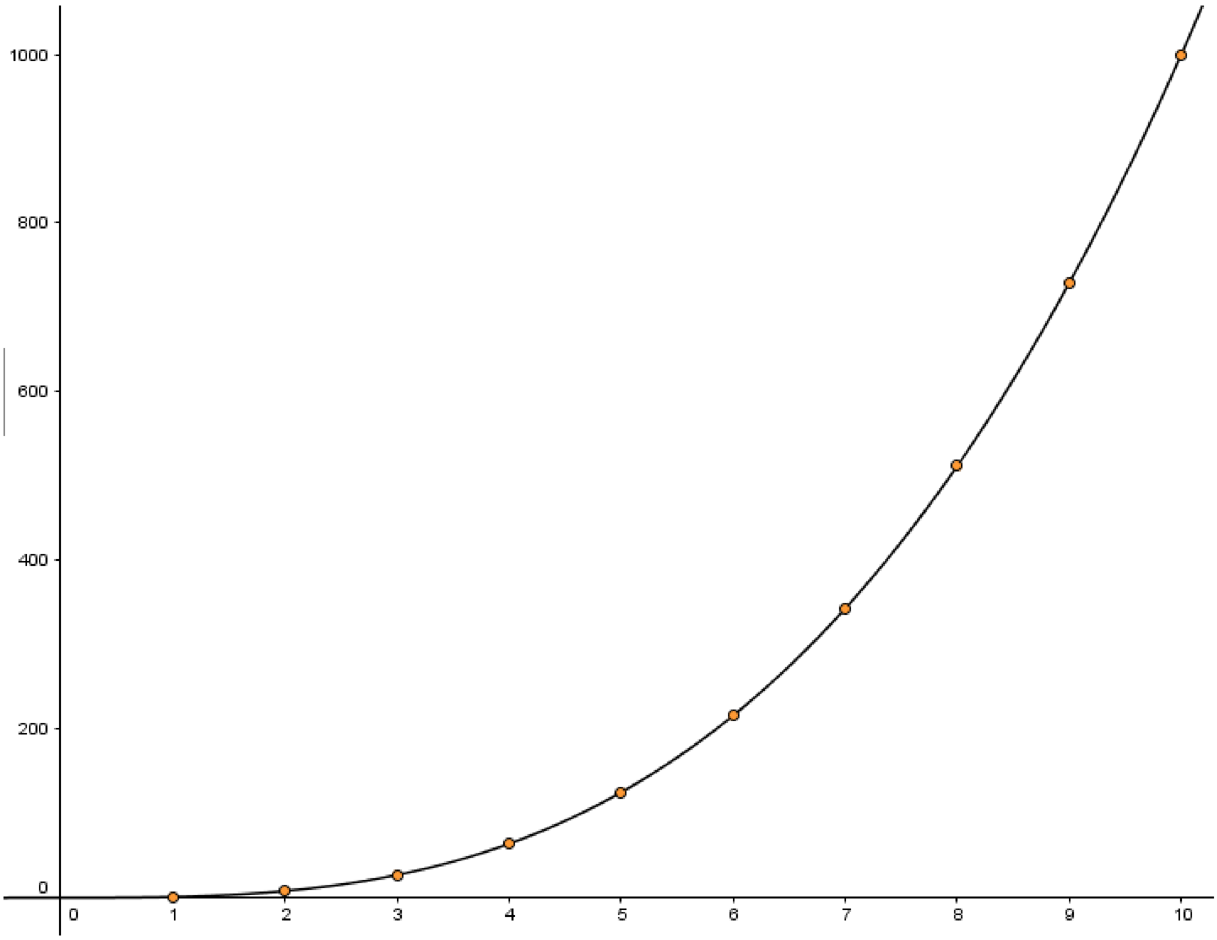


Image 10

The formula for the number of small cubes with zero visible sides:

$a = x^3$

We show the relationship in a coordinate system. The line is the graph for our formula $a = x^3$ and the points represent the number of small cubes with zero visible sides. This confirms that the formula works for all figures as all the points are on the curve.



Graph 1

One visible side

The small cubes with one visible side are on the six sides of the figure, see *Image 11*. On every side, they form a square cuboid. The small cubes that make up the edges have either two or three visible sides. This means that the sides of the square cuboids are two less than the whole figure's side. The cuboid's side is $(x + 2)-2 = x$. We calculate the area by multiplying the sides by each other, $x \cdot x = x^2$. Then we multiply it by six, because there are six sides.

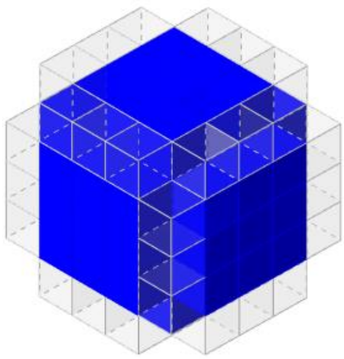
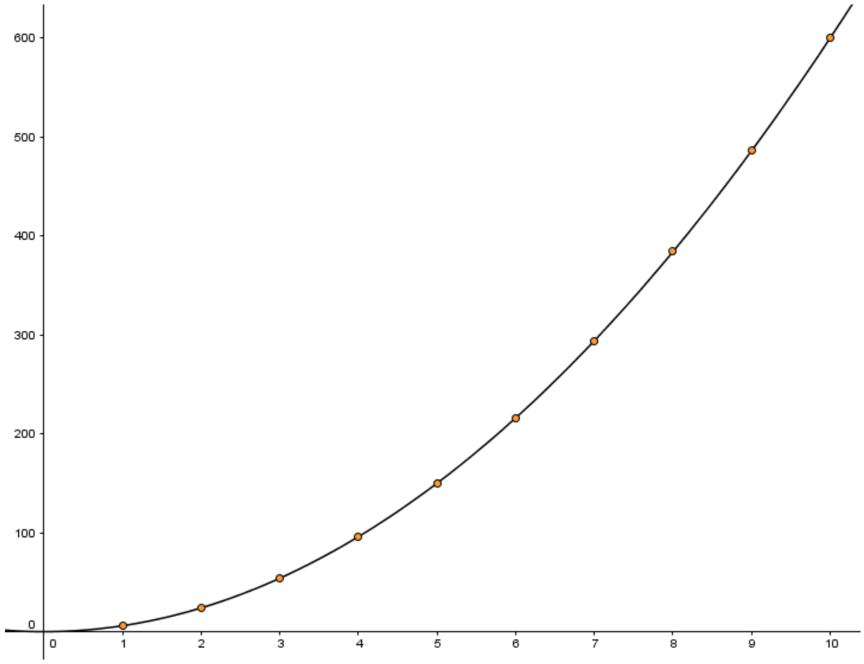


Image 11

The formula for the number of small cubes with one visible side:

$b = 6x^2$

We show this curve in *Graph 2* and see that the formula works for all figures as all the points are on the curve.



Graph 2

Two visible sides

The small cubes with two visible sides are placed in the middle of the twelve edges of the image, *Image 12*. We cannot count the whole edge, as the outermost small cubes have three visible sides. There are two cubes with three visible sides on each edge, and each edge is made up of x small cubes. Therefore there are $(x – 2)$ small cubes with two visible sides on each edge. As there are 12 edges we multiply by 12.

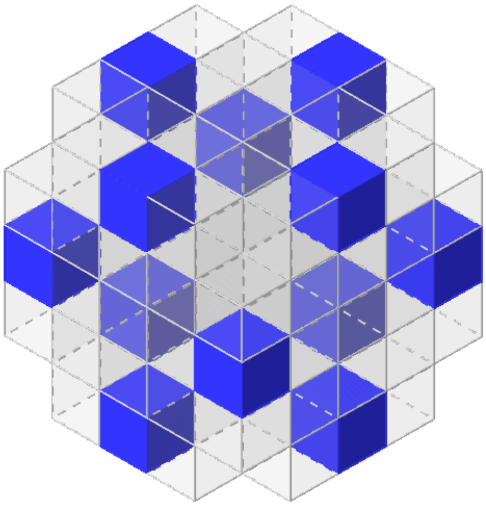


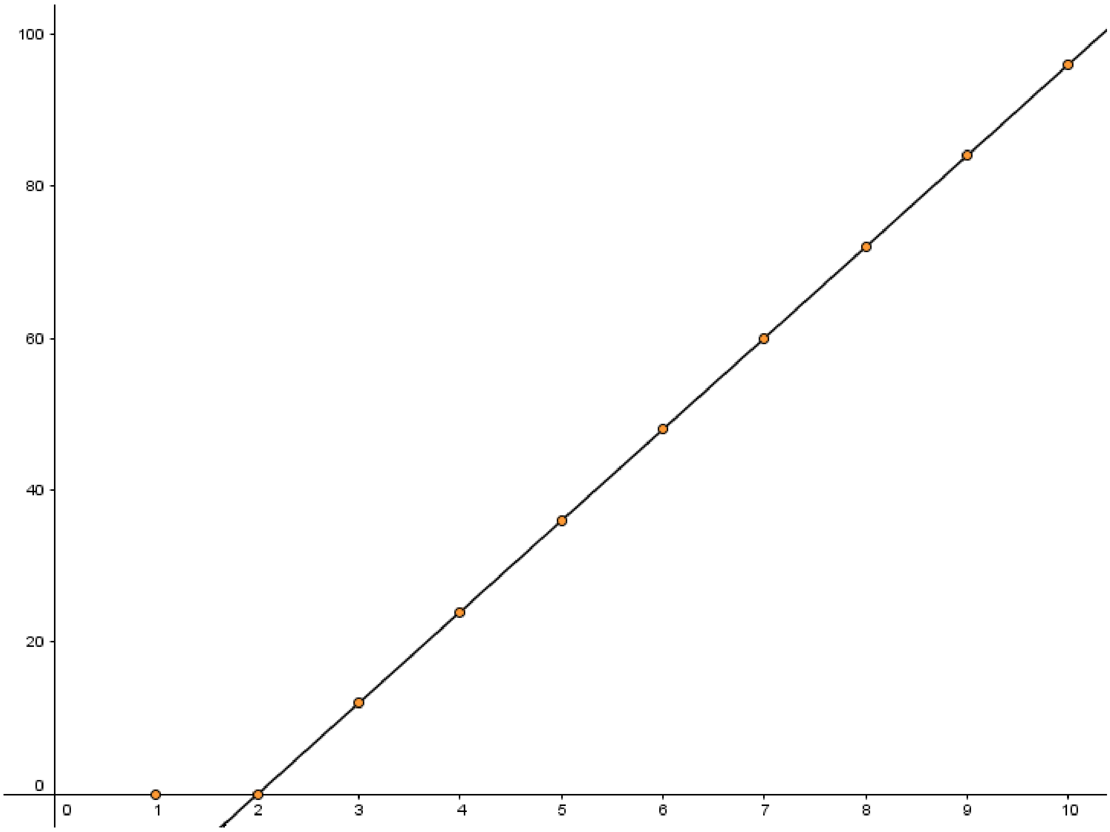
Image 12

The formula for the number of small cubes with two visible sides:

$c = 12(x – 2) = 12x - 24$

The formula does not work on Figure 1, since we would get a negative number of cubes, which is unreasonable. To avoid this problem, we regard it as an exception and put the negative number of cubes as zero.

In *Graph 3* we show how the formula works for the first few figures and see that it does not work for Figure 1, as the first point is not positioned on the line.



Graph 3

Three visible sides

The small cubes with three visible sides are the corners in the image, *Image 13*. They sit in eight groups of three. There are no small cubes with three visible sides in *Image 1* and therefore the formula does not apply for that image.

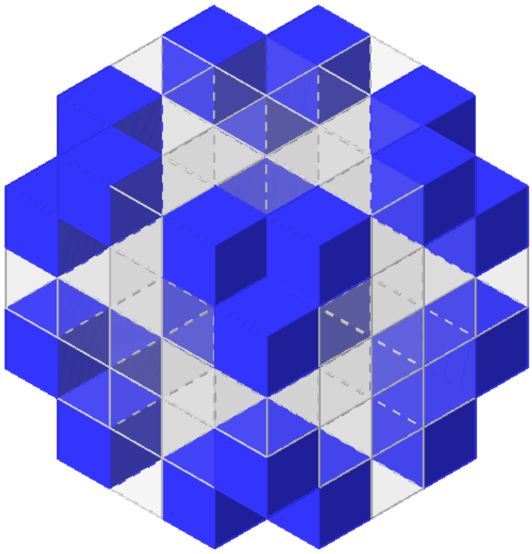


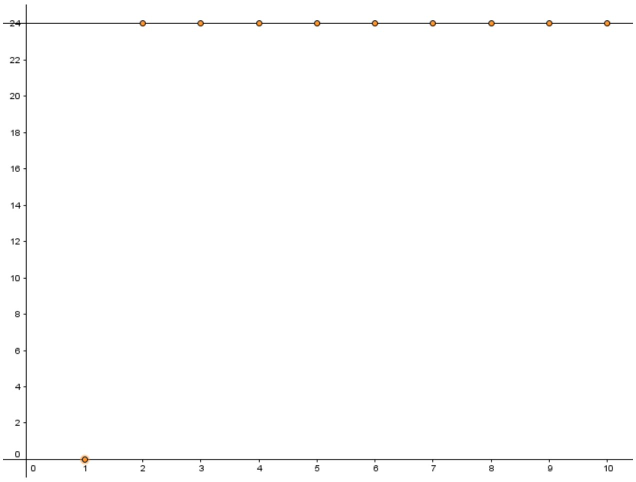
Image 13

The formula for the number of small cubes with three visible sides:

$d = 3 \cdot 8 = 24$

We can also see that there are two small cubes with three visible sides on each edge, and get the formula $d = 2 \cdot 12$, because there are 12 edges.

Here we also show the relationship with a graph and again we see that the formula does not work for Figure 1, as the first point is not positioned on the line, but it still works for the rest of the figures.



Graph 4

Four visible sides

The small cubes with four visible sides we only find in *Image 1*. There they are placed on the edges, *Image 14*. There is only one cube on each edge, and the figure has twelve edges.

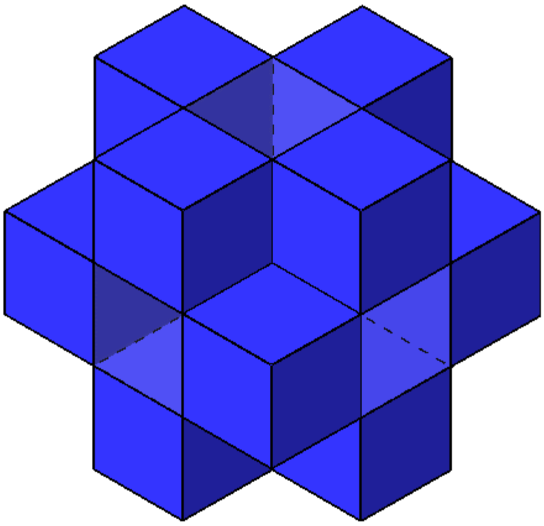


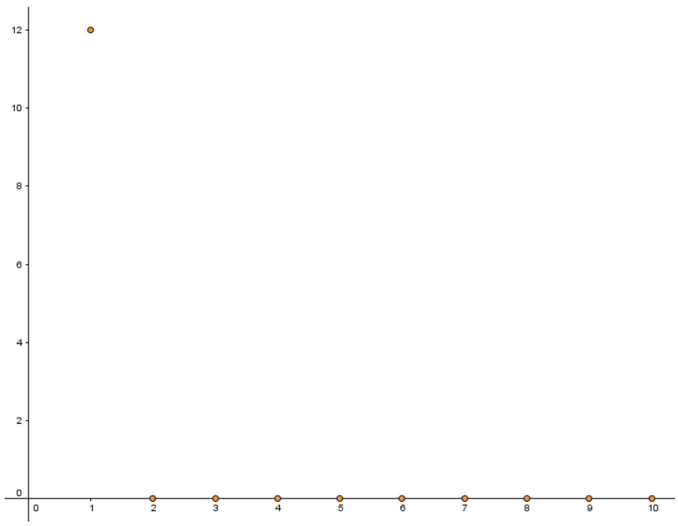
Image 14

The formula for the number of small cubes with four visible sides:

$e = 1 \cdot 12 = 12$

The formula applies only to figure 1. No other figure has small cubes with four visible sides, instead they have small cubes with two and three visible sides on the edges.

The relationship can be seen in the graph below. As only Figure 1 has any cubes with four visible sides it is the only point not positioned on the line $e = 0$.



Graph 5

Image's number	**Number of small cubes with: 0 visible sides**	**1 visible side**	**2 visible sides**	**3 visible sides**	**4 visible sides**	**Total amount of small cubes**
1	1	6	0	0	12	19
2	8	24	0	24	0	56
3	27	54	12	24	0	117
4	64	96	24	24	0	208
5	125	150	36	24	0	335
6	216	216	48	24	0	504
7	343	294	60	24	0	721

Table 4

In *Table 4* we can again confirm that all small cubes with 0, 1, 2, 3 and 4 visible sides together is the total number of small cubes.

Relationships

When we compare the formula for small cubes with different number of visible sides with the formula for the total number of small cubes in the figure, we see an obvious relationship. There are x^3 small cubes with zero visible sides in all of the figures, and this expression we find in the overall expression for the entire figure as well, $y = x^3 + 6x^2 + 12x$. The expression $6x^2$ we also find in the formula of the number of small cubes in the entire figure. $12x$ we find in the formula for small cubes with two visible sides. In that formula we also find -24. This is corrected as the formula for the number of small cubes with three visible sides is 24. 24 and -24 cancel each other out. Each small cube has either zero, one, two, three or four visible sides. As a result we get

$a + b + c + d + e = y$

Visible sides	Expressions	Exceptions
0	x^3	
1	$6x^2$	
2	$12x - 24$	Figure 1
3	24	Figure 1
4	0	Figure 1
0, 1, 2 and 3	x^3+6x^2+12x	

Table 5

$y = x^3 + 6x^2 + 12x - 24 + 24 + 0 = x^3 + 6x^2 + 12x$

Conclusions

The group worked with a pattern of a growing figure. The goal was to solve the assignment with as many different methods as possible, and compare them with each other. We also tried to find relationships between the solutions.

We started by using different methods to find solutions to the assignment. The solutions were divided into five different categories depending on the method used. We chose the most suitable solutions to represent their categories in the report. In each solution, we worked further in order to describe our thoughts in an understandable way using tables, formula, texts, images and graphs.

One of the other tasks was to find the expression for the number of small cubes with zero, one, two, three or four visible sides. After we solved this task we were able to conclude that all formulas for small cubes with different number of visible sides together made up the formula for the total number of small cubes in the entire figure.

Thanks to this assignment, we now have new mathematical knowledge when it comes to simplifying complex expressions, for example $(x + 2)^3$, and the whole group has learned how to cooperate better and to take advantage of each individual's skills.

References

Books:

- Susanne Gennow, Ing-Mari Gustafsson och Bo Silborn (2011). *Matematik för gymnasiet Exponent 1c.* Malmö: Gleerups.
- Anker Tiedermann. (1999). *Talen magi. Lustläsning för talfreaks.* Stockholm: Berghs förlag AB.
- Christer Kiselman, Lars Mouwitz (2008). *Matematiktermer för skolan.* Göteborg: Livréna AB.
- Harold R. Jacobs (1970). *Mathematics: A human endeavor.* USA: W. H. Freeman & Company.

People:

- Erland Arctaedius
- Johan Thorbjörnson

Log Täby Friskola

FEB

8

When our teacher Anna tells us that we have advanced to the semifinal in Sigma8, everyone in the Math group scream with joy! We immediately start looking at the assignment that we are going to work with. In pairs we study the pattern of 3D figures. Every one writes down their thoughts.

10

We continue solving part A in the assignment. We have earlier worked a lot with patterns, but not in 3D, therefore this will be a fun challenge. Our group comes up with several solutions. Many of us get a formula of third degree. Together with Anna we learn how to simplify such a formula, since we don't have any previous experience of it.

16

We begin the lesson by going through our solutions. We pick the ones we would like to work with. After that some students are elected to work with the report. They start writing the solutions in a document. After that the rest of us get the task of finding out the meaning of the word *congruent reproductions*. We ask teachers, parents and search for facts on the internet. We find that two figures are congruent if they have the exact same angles and measures.

One student spots some pictures in the classroom. Since they consist of congruent, geometric shapes, we ask Anna what they are. She says that they are pictures made with the same technique as the famous artist Escher used. He created art from mathematics.

We also look for buildings with congruent architecture. Some students think that buildings aren't quite congruent and would prefer a more exact example. One of the suggestions is DNA.

25

Today the mathematician Johan Thorbiörnson visits us. He gives an inspiring lecture where we learn more about congruent figures. The group decides what kind of congruence example to use. The suggestions are buildings, DNA and art by Escher. We give our votes and DNA is the winner by far. Johan thinks it is a fun idea, and calls it “Escher 2.0”.

MAR

8

We start the lesson with electing the four students who will represent us in the semifinal. Several of us are interested and the voting is even. After that we divide into groups and the report group continues writing, while the others search for facts about DNA in books and on the internet. The cooperation in the report group doesn’t work as well as expected and the technique failing doesn’t make it any better. The group is too big, which makes it hard to agree.

9

Today the report group is smaller and luckily the cooperation and the technique work. Some students are searching for qualified people with deep knowledge within the biomedical area. The rest of us develop our knowledge about congruence.

14

Per Johan Råsmark, educated biomedical engineer, visits us. He gives a presentation on congruence and DNA. He helps us understand the difficult words that we can’t get understandable explanations for. In order for something to be congruent it must be completely overlapped by reflection, rotation or transferring. In the end of the lesson, we make a terrible detection! If we reflect the DNA molecule, the twisting of the spiral goes the other direction, which makes it lose its function in our world. Is DNA not congruent?

There is silence for a few seconds, and then a wild discussion starts. What are we going to do now? Do we go on with DNA or start with something new? It feels like a failure to give up now.

Now we’re in a hurry! We contact Johan Thorbiörnson for guidance. The day after that, he accidently runs into Mathias Uhlén, one of Sweden’s most prominent researchers and a professor of microbiology. Together they discuss whether the DNA molecule is congruent or not.

16

We decide to go deeper into the understanding of congruence and how it is linked with the DNA molecule. A student called Serhat Aktay from KTH, The Royal University of Technical Science, visits us and explains how the reflection affects the function of the DNA. We also plan how we want our exhibition to look.

APR

4

Thanks to a student’s parent we’ve managed to arrange an interview with Mathias Uhlén. We ask him if he considers the DNA molecule congruent with its reflection. His answer is no, but after some discussion we find that it depends on how you interpret the definition of congruence. It is interesting to hear a professor’s thoughts on the matter and he encourages us to complete our work.

8

This period has been both challenging and worthwhile. Working with a bigger project over an extended period of limited time, meant that we had to be responsible and use the time given. We also learnt to follow specified criteria for this assignment and use external competence to understand. We’ve overcome difficulties without losing our enthusiasm. By dividing into smaller groups, focusing on different areas, everybody has got their chance to shine. We have been dependent on each other and our friendship has strengthened. Everybody has taken an equal part and done their best, which has led to a result we’re very proud of.