Nordic Math Class Competition 2017

Choice of routes in a grid

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Number of characters: 11.313
Appendix number of characters: 2.554
Total number of characters: 13.867

Contents

Introduction

We have qualified for the finals in the Nordic Math Class Competition where we have been faced with the assignment *Choice of routes in a grid*. We were given a paper with a 5 * 6 grid of squares. On this grid the points S and M were located on their own grid point. We had to investigate how many different routes we could take from the starting point S to the end point M. But there were some rules we had to follow:

Rule I was that we could move along the horizontal lines that S and M were located on, and on all the lines between them.

Rule II was that we could move along all the vertical lines.

Rule III was that we couldn't cover the same path twice within one route.

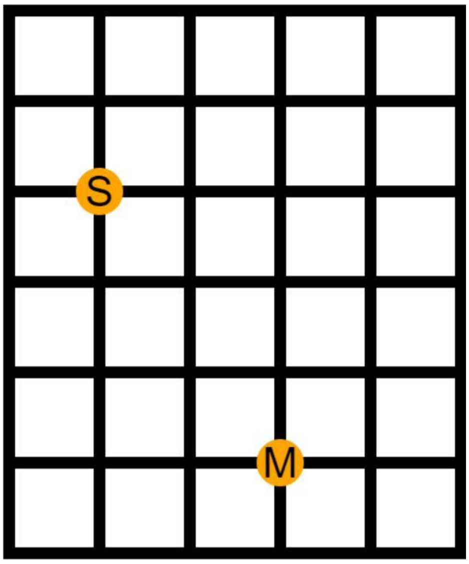
And finally Rule IV was that we could not go back to a horizontal line when we already had left it.

Exhibition

Based on the given problem, we chose to make our exhibition a big city with a taxicab driving from location to location. The routes represent the lines of the grid, and the buildings represent the squares of the grid. We think it's a good way to show the problem in an interesting way. Like getting from the starting point to the end point in the problem, there's no reason to drive away from the destination, because that would be a waste of money and time.

We also thought that we could place road signs that would show where you could and couldn't drive. For example could a road sign show that the road was unidirectional.

Choice of routes in a grid



1) How many possible routes are there from S to M in the above figure?

We need to move from the starting point S to ending point M. The start point is in this case located over the end point, so therefore we have to move downwards to get there. Because moving upwards would require us to cross one of the horizontal lines that we have already moved along or crossed, and that is forbidden by rule IV. Therefore we have to move downwards.

First we have to move from the start point's horizontal line down to the line right below it. To get down we need to move along the vertical lines. No rules forbid us from moving all the way out the outermost vertical lines, so therefore we can move down using all of the grid's six vertical lines. We have now moved down to the line below the starting line, were we have the same choice as before; six possible routes to the line below. On the next line there are again six possible routes, and afterward we have reached the horizontal line with the end point. It doesn't matter if we have reached it using a vertical line to the left, right or above the end point; we now only have one possible route to the end point.

To find the number of possible routes from S to M, we now have to calculate the number of all possible combinations of the possible routes on each horizontal line. On the horizontal line with the starting point we had six possible routes, the same for the line under and the line under that, and on the last line we had only one possible route.

That means we can use the multiplication principle in combinatorics to calculate the number of possible routes by multiplying the first six possible routes by the next six, the next six again and finally by one. The last single possible route will always occur no matter what grid we are using, and multiplying by one doesn't change the number so the last step is unnecessary.

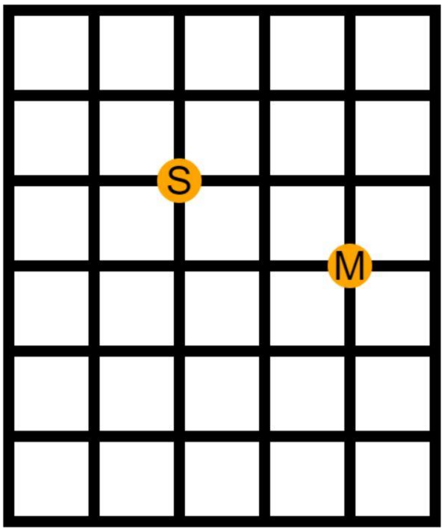
Therefore we can calculate the number of possible routes with $6 * 6 * 6$, which also can be written as 6^3. The number of possible routes is then <u>216</u>.

2) Choose different start and end points and find out how many possible routes there are between them.

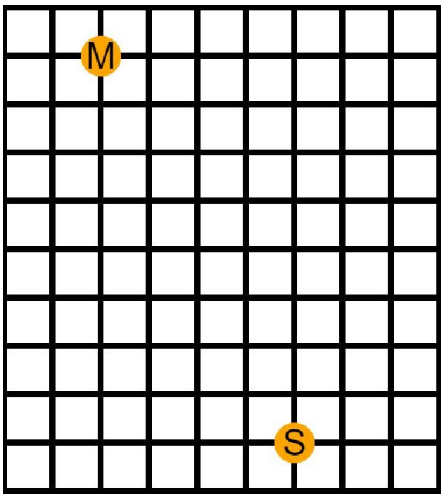
The calculation of the result that we came to before was 6^3. 6 was the number of vertical lines, and 3 was the number of horizontal lines that we had to move past.

We had to move one line down for every square on the y-axis between the start and end point. That means we can make a formula called V^D where V is the number of vertical lines and D is the difference between the points' y-coordinates. This formula will from now on only be referred to as V^D.

The formula V^{D-1} can also be used, where V is the number of vertical lines and D is the number of lines between the points, including the lines that the points are located on. This formula is basically the same because we get the same numbers, so we just chose to use the formula V^D because it is shorter.



If we choose other start and end points we can now use that formula. From now on we have chosen to place the origin point in the bottom left corner. Every section is from the 1st quadrant in the Cartesian coordinate system. If we choose the points S(2,4) and M(4,3) on the same grid as before, then the difference between their y-coordinates is $4 - 3 = 1$. Therefore the formula would become 6^1 and the number of possible routes would be <u>6</u>.



We can for example also choose the points S(6,1) and M(2,9) on a 9 x 10 grid. Here we need to move upwards to get to the end point, and not downwards like in the other examples. If we would first move down and away from the end point then it would again be impossible for us to get to the end point, which in this case is positioned above.

We can still use the same formula as before. The difference between these points' y-coordinates is $9 - 1 = 8$, and this grid has 10 vertical lines. Now the formula will be 10^8 so the number of possible routes is <u>100.000.000</u>. The formula is exponential so therefore the number of routes will quickly become very large.

We also investigated whether or not it was possible to use Pascal's triangle to calculate the number of possible routes.

We were not able to find a possible pattern in the triangle for our specific assignment, but if we change some of the rules, then it is possible:

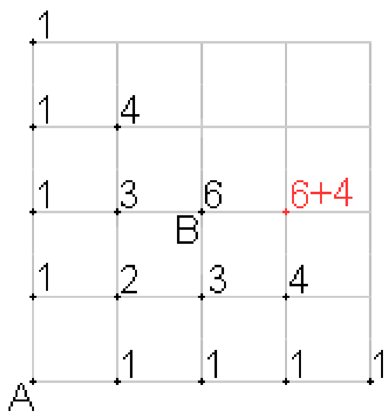
Rule I needs to become that we can move along the horizontal and the vertical lines that S and M are located on, and on all the lines between them.

Rule II will no longer be relevant.

Rule III will still be that we can’t cover the same path twice within one route.

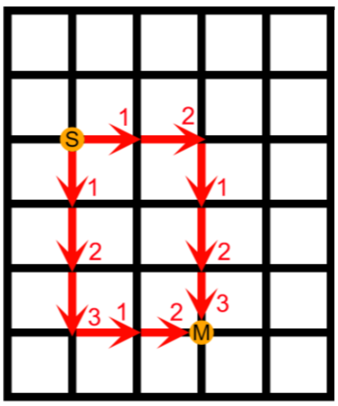
And finally Rule IV needs to be that we can not go back to a horizontal or a vertical line when we have already left it.

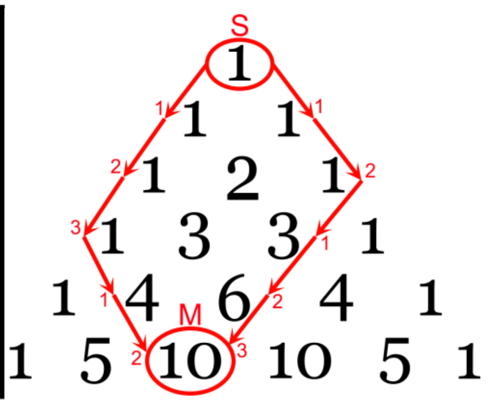
If we have to get from the starting point S to the end point M in our original grid using these new rules, there are significantly less possible routes because we now also can’t move backwards horizontally.



If we count the possible routes to the first closest end points, we can write the numbers into our grid. We now start seeing a pattern that is Pascal’s triangle - just rotated and laid over our square grid.

Therefore we can also find this number by using Pascal’s triangle, where we place S on the number in the top of the triangle. On the grid we have to move 2 squares in one way and 3 squares in the other way to get to the end point. This means that we need to move 2 numbers down one of the sides in the triangle, and then 3 numbers down the other side. We now end up at the number 10, meaning there are 10 possible ways from S to M using these new rules.





We can also put the number of possible routes using the original rules into a grid. It doesn't have a pattern similar to the one we just showed you using the other rules; hence we were not able to find this pattern in Pascal's triangle.

1	1	1	1	1
6	6	6	6	6
36	36	36	36	36
216	216	216	216	216
1296	1296	1296	1296	1296

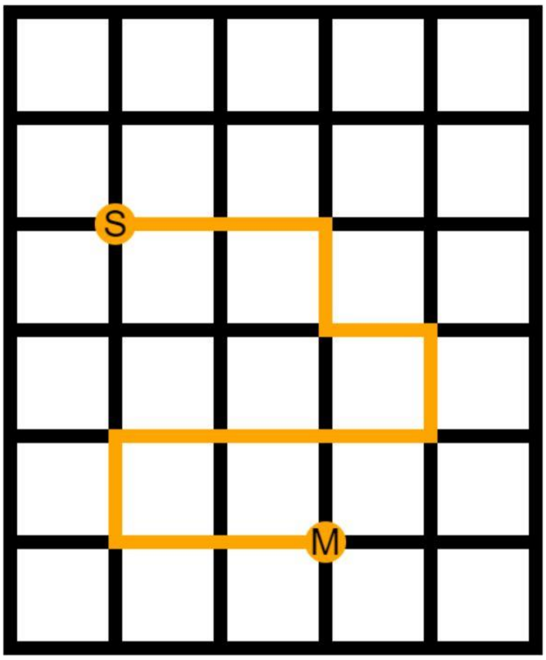
1	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70

3) Make a rule that shows how many possible routes there are between two points on an infinitely large grid.

Our formula is as said V^D.

In an infinitely large grid the formula will become ∞^D. So it doesn't matter where the points are placed or what the difference between their y-coordinates is, the number of possible routes will always be infinite.

4) Find at least two ways of describing a route between two points.



We can describe a route with a picture of the grid, where one specific possible route between the start point and the end point is highlighted. On the picture to the right a single route between the points on our original 5 * 6 grid is highlighted in orange. In this way we can clearly describe all the 216 possible routes between the points.

We have also made a website which shows a custom grid with a custom start and end point. We have programmed the website to show all the possible routes between the points as an animation, played at a custom speed. The website can be found at http://bit.ly/nmcc-denmark

We can also describe a route with numbers. On each line where we have to move up or down to reach the end point, we can move along any of the vertical lines. That means we can describe all our choices in a single route between the points as a series of numbers.

The first number in the series corresponds to the first choice in our route: It is the x-coordinate of the vertical line we chose to move along, from the horizontal line with the start point and to the line right above or under in the direction of the end point. The second number corresponds to the x-coordinate of the vertical line we chose to move along in our second choice in the route to the end point, and so on.

If we with a series of numbers had to describe the same route that we described with a picture before, it would be 3, 4, 1.

Conclusion

Summarizing our results

We began by thinking about what options we had to move on the lines in the grid. By using the multiplication principle in combinatorics we got the result of 216 possible routes from the starting point to the end point. Then we generalized the method by making the formula V^D , that can be used to calculate the number of routes in an arbitrary grid between any two points. To visualize the number of routes we made a website, that we programmed to show the grid with all the routes highlighted as an animation.

The work with the assignment

When we worked on the assignment for NMCC we split our math team into small groups, who each focused on different parts of the assignment. By splitting the team, the assignment got more manageable because the groups were focusing on a specific part of the bigger project. To make sure that we all knew what everyone were doing, we informed each other of our results at the end of each lesson.

The first time all the groups worked on the grid problem and suggested solutions. Afterwards each group presented their solutions. We talked about which solution was the right one, and in the end we all agreed on an answer.

After this we split our class into new groups, with each group working on different assignments. Some were working on combinatorics, others were designing our exhibition, preparing our subject report or preparing our presentation.

Our teacher Katrine has been helpful when we demanded further explanation of the assignments if there was something we didn't understand. When people finished their assignments they started working on new investigations, which maybe could be relevant for our subject report.

Social and academic development

While working on this project we have developed ourselves academically and socially. We have learnt a lot about how to make a mathematical subject report while working on this report.

We have gathered new mathematical knowledge through our work with combinatorics and Pascal's triangle.

By working on our exhibition we have increased our creative abilities.

When Katrine wasn't allowed to help us with something, we had to get help from each other which has had a positive influence on our team.

Our work has been very free, because we could freely choose how we wanted to work with the assignment.

Logbook

On Monday 03.06.2017

Today we got the news that we're going to the Danish finals in the Nordic Math Class Competition. Our local newspaper came to visit us to ask us some questions and take some pictures of us. Our teacher Katrine told us a lot about the contest. Afterwards we all started working on the assignment *Choice of routes in a grid*. We presented our own answers and agreed on our solution.

On Thursday 03.09.2017

Today we chose the four participants who was going to represent us in the contest. Afterwards some began work on the subject report, while the rest worked on the exhibition.
In the end of the lesson the exhibition team presented some of their ideas. We voted on the idea for the exhibition we liked the most. We chose to make a city like New York.

On Monday 03.13.2017

Today we continued working on our individual assignments. The exhibition team decided what the city should look like and which buildings should be a part of it. They gathered the materials and planned the making of it.
The report team wrote the introduction to the report and continued working on the problem assignments.

On Thursday 03.16.2017

Today half of the report team continued work on the problem assignments, and the other half began checking the report.
The exhibition team made some of the houses for the city, and thought about how to incorporate the math into the exhibition. Some people didn't have anything to do so they began studying taxicab geometry, hoping it could be relevant for the subject report.

On Friday 03.17.2017

Today we continued our work from the last time. The exhibition team started on the platform for the city, and they also made some more buildings. Some of the others studied if there was a connection between Pascal's Triangle, taxicab geometry and combinatorics.

On Monday 03.20.2017

Today we again worked in the different groups. One of the groups continued to study the connection between taxicab geometry and Pascal's Triangle. The exhibition team made even more buildings for the exhibition. All of our logbooks also got collected in one document.

On Thursday 03.23.2017

Today most of the groups continued their work, but some of us began working on something new. Some started work on the conclusion of the subject report, while others read the report and corrected mistakes.

On Friday 03.24.2017

Today we finished our subject report and logbook.

The exhibition team also got close the finishing their work by making the last buildings and drawing the lines for the roads. Some also began writing some text for the exhibition.