Locus

NMCC 2021

Täby Friskola Sweden

Content

Introduction

When we found out that the Sigma 8-finals were going to happen despite of Covid-19, we were thrilled. We had heard of the experiences of previous students and were afraid that we would miss out on them. The subject matter Locus was challenging. We had never heard the word before. However, once we began studying the subject, our interest was piqued. It proved to be a new way for us to conceptualize geometry.

For part of the project, we at Enhagen have worked from home, which was a bit of an endeavour. Our ability to communicate was limited. It was difficult to move the project forward. To simplify cooperation, we were randomly split into small groups using video calls. Everyone contributed with solutions to the assignment in Part 1 and helped provide drafts for Parts 2 and 3. Three groups, based on interests and abilities, then focused on specific parts of the assignments. We elaborated and summarized everyone’s suggestions. Beyond regular math-classes, we worked solely on the project for one week to finish on time.

Petter's House

We are to find all points where Petter's house (H) can be placed and determine the Locus. The distance between the train station (T) and city hall (R) is 300 m. H is three times as far away from R as from T. We used GeoGebra to create images of the solutions.

We place R in the origin point in a coordinate system. T is located 300 meters from R, T= (300; 0). We notice 2 possible points for H, both in line with R and T. These points are marked H_1 and H_2. H_1 is 225 m from R and 75 m from T, H_1= (225; 0). H_2 is placed 150 m from T and 450 m from R, H_2= (450; 0). Both H_1 and H_2 are 3 times further from R than from T.

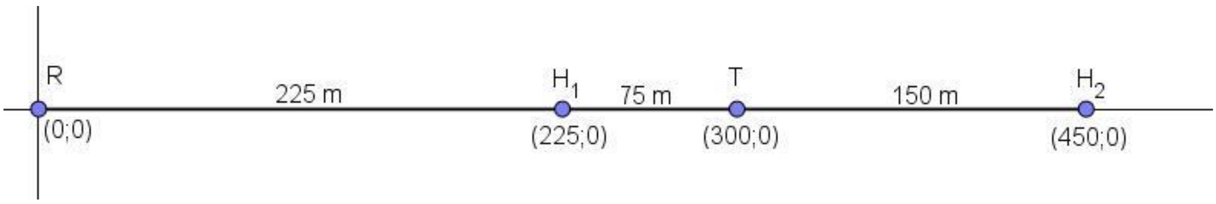


Fig. 1

However, H does *not* have to be aligned with T and R. We make a right-angled triangle with R and T in a corner each. H_3 is placed straight above T.

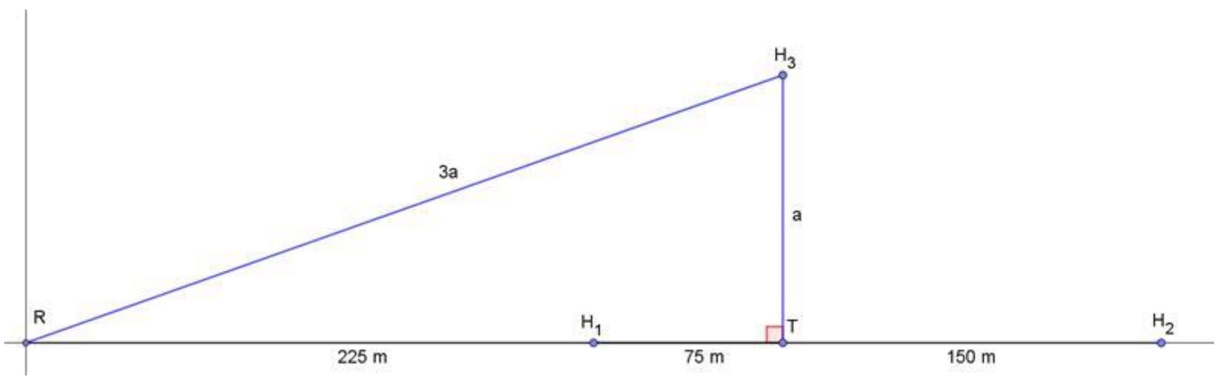


Fig. 2

According to the Pythagorean Theorem:

$$(3a)^2 = a^2 + 300^2$$
$$8a^2 = 90\,000$$
$$a^2 = \frac{90\,000}{8}$$
$$a = \frac{300}{\sqrt{8}} \approx 106.1$$

This gives us H_3= (300; 106.1).

We draw a circle where H_1, H_2 and H_3 are placed on the periphery. The distance between H_1 and H_2 is the diameter (d):

$$\text{d}=|H_1T| + |TH_2| = 75 + 150 = 225\ m$$

The diameter is 225 m. The circle centre is 112.5 m from H_1 and H_2, the centre is in (337.5;0) (Fig. 3). We find several possible locations for H. All points on the circle periphery represent Locus for H. H_1, H_2 and H_3 are placed 3 times further away from R than from T. The relation is 3:1.

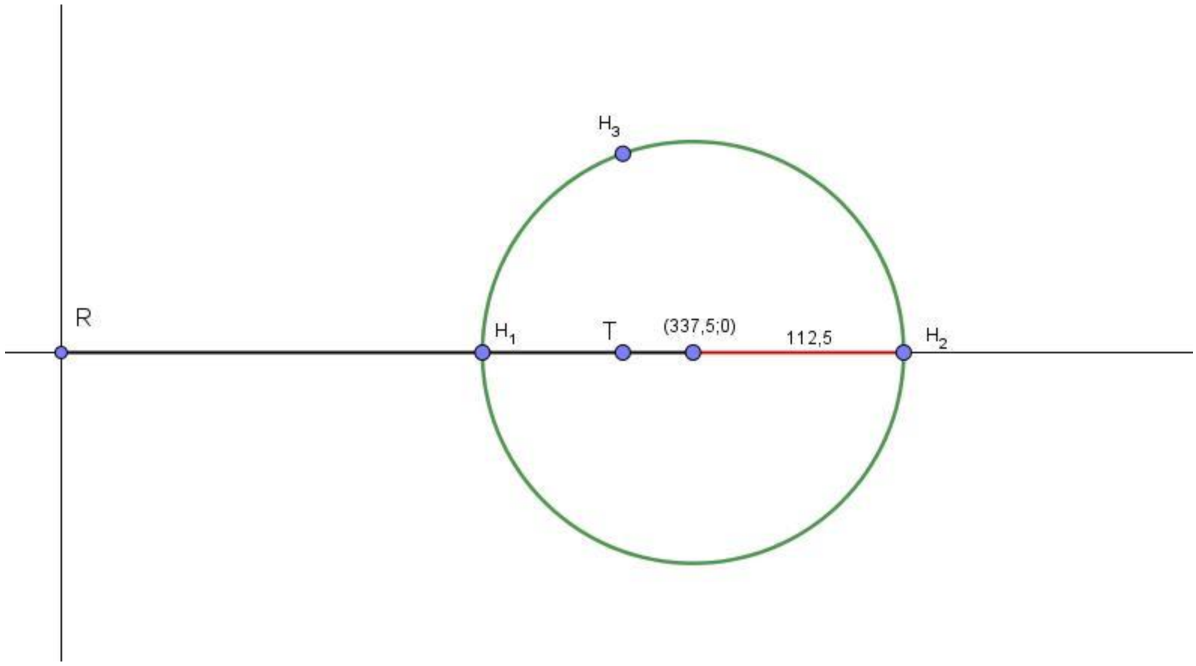


Fig. 3

The challenge is to prove that the relation 3:1 is true regardless of where on the periphery H is placed. One student, Erik, proposes that we draw all possible points on the periphery. This suggestion is rejected, as there may be an infinite number of points.

Another student, Lena, suggests “Can we prove our Locus theory using the Pythagorean Theorem the same way we proved house 3?” Some of us think this is possible, others are confused. Our teacher, Anna, feels that everyone should be included. We spend some time understanding Lena’s idea and cooperate to formulate the mathematical proof.

Proof

We select an arbitrary point, coordinates (x;y) and determine $|RH|$ and $|HT|$ using the Pythagorean Theorem.

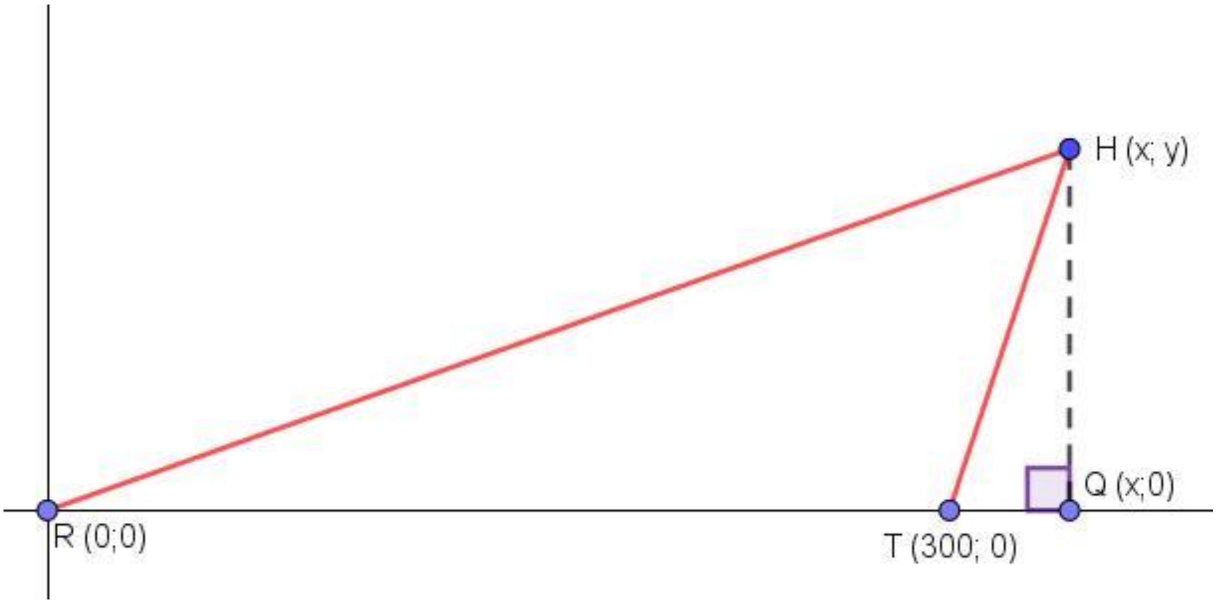


Fig. 4

$$|RQ| = x \text{ m}$$
$$|HQ| = y \text{ m}$$
$$H = (x; y)$$
$$|RT| = 300 \text{ m}$$
$$|TQ| = (x - 300 \text{ m})$$

ΔRHQ and ΔTHQ have straight angles.

$$|RH| = \sqrt{x^2 + y^2}$$
$$|TH| = \sqrt{y^2 + (x - 300)^2}$$
$$3{:}\,1 = \sqrt{x^2 + y^2}{:}\sqrt{y^2 + x^2 - 600x + 90\,000}$$
$$(x^2 + y^2){:}\,(y^2 + x^2 - 600x + 90\,000) = 9{:}\,1$$
$$x^2 + y^2 = 9y^2 + 9x^2 - 5400x + 810\,000$$
$$0 = 8y^2 + 8x^2 - 5400x + 810\,000$$

The formula is difficult to interpret. We feed it into GeoGebra and get this figure:

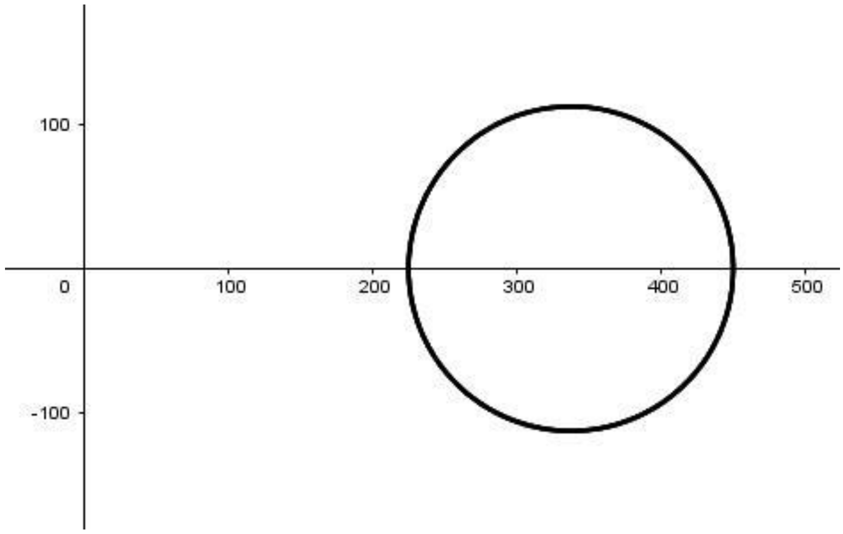


Fig. 5

A circle!

GeoGebra transforms the formula:

$$(x - 337.5)^2 + y^2 = 12\ 656.25$$

We recognize the value 337.5 as we got it earlier when calculating the circle centre. The circle *is* the Locus for H. The generic formula for a circle is:

$$(x - a)^2 + (y - b)^2 = r^2$$

Which means that 12 656.25 m is the circle radius squared.

$$r = \sqrt{12\ 656.25} = 112.5\ m$$

This proves that Locus is a circle with the centre at (337.5; 0) and the radius 112.5 is correct since we used an arbitrary point.

We struggled with the equations. When we realized that we had managed to work out the Locus, we felt euforic. In Part 1, we learned about the new concept of Locus. It gave us an opportunity to further explore Locus in Part 2.

New Problems

Randomized into groups of four, we find inspiration to new problems online. As our class is multilingual, we find several sources and create three problems with the right difficulty.

The Log and the Beaver

A beaver (B) lives in a dam. It aims to collect the longest possible log without getting it stuck in a 6-meter-wide channel. The channel has a straight angle turn and parallel sides. The beaver always sits at the same spot on the thin, floating log. It sits in accordance with the Fig. below, in a 1:5 ratio:

Fig. 6

The Locus of the Log

The length of the log, needed to find the Locus, is modelled in GeoGebra. We draw the channel sides on the coordinate axes and two lines parallel with them. We see that the longest possible log that can travel through the channel is 16.97 m.

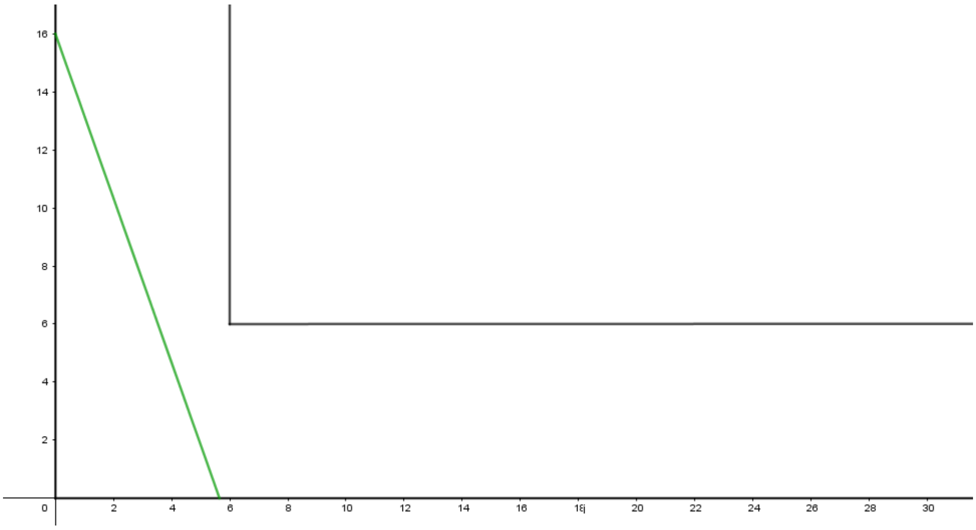


Fig. 7

We draw lines with the length of 16.97 m to show the log's movement through the channel.

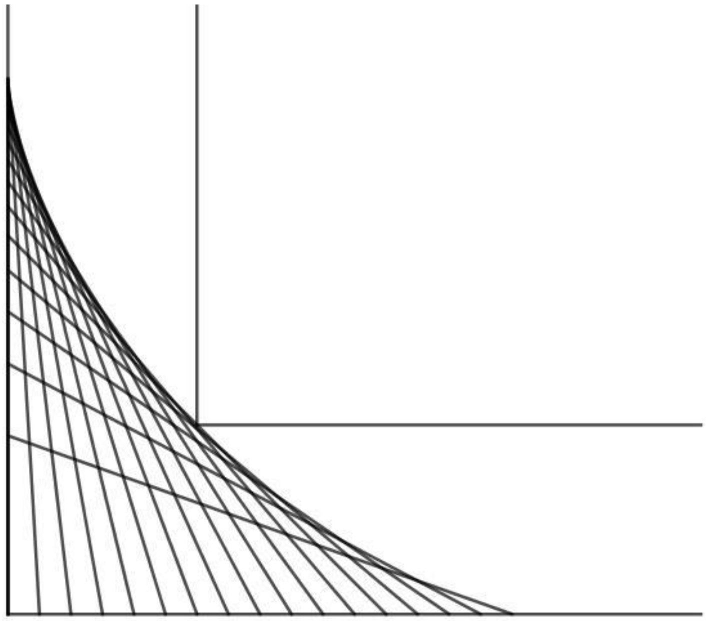


Fig. 8

We get the log's Locus by creating an animation with the distance 16.97 m. The yellow line is the log and the blue area is the Locus of the log (Fig. 9).

Press here to see the animation.

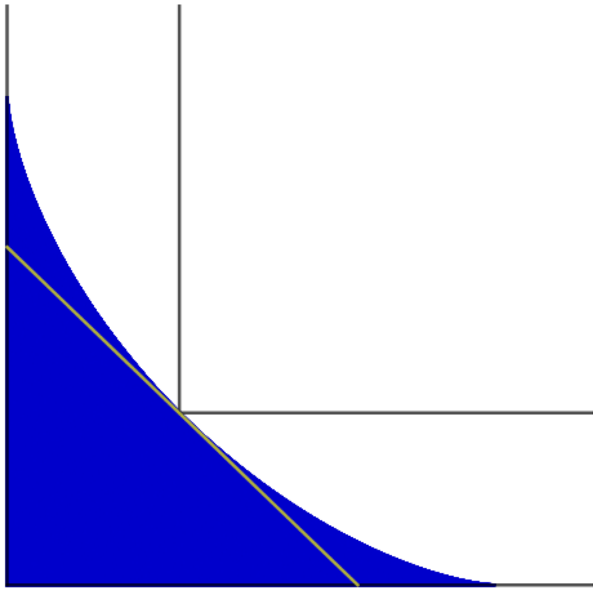


Fig. 9

We see that the Locus border traces part of an *astroid*, a sort of *hypocycloid.* An astroid is the Locus for a point on a circle periphery, rolling inside another larger circle where the relationship between the circles' radii is 1:4 (Fig. 10). Press here

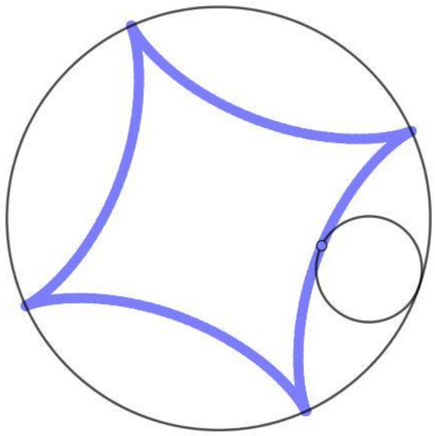


Fig. 10

We find a general formula for astroids online. We use it to calculate the length of the log in Fig. 9 and receive almost the same length as when we modelled it in GeoGebra. We did not include this formula in the report. It is complicated and only understood by a few of us. The investigation showed that the Locus of the log is $\frac{1}{4}$ of a filled astroid. The length of the log is therefore the radius of the astroid-circle.

The Locus of the Beaver

We investigate the beaver's (B) Locus as it is sitting on the log travelling in the channel. We use the geometric ort tool and our previous knowledge of the log movement to calculate B Locus. Press here

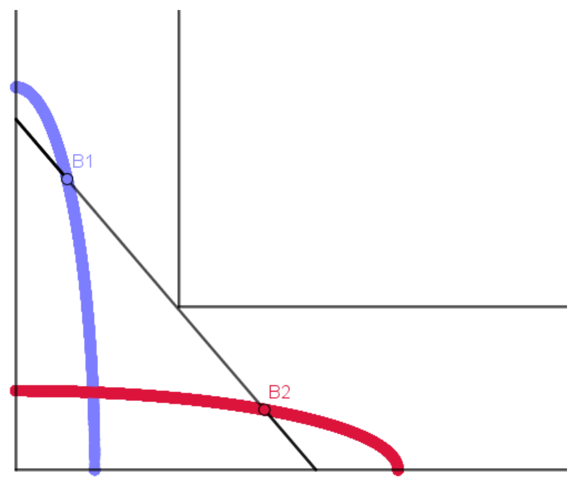


Fig. 11

Fig. 11 shows the Locus to be $\frac{1}{4}$ of an ellipse. The Locus can take form in two shapes depending on whether the beaver is located on the fore or aft section of the log.

The Gathering of Bees

Oscar's bee

A queen and her three worker bees live in a garden. The garden has three congruent, circular equidistant flowerbeds. Each bee circulates its own flowerbed periphery. The queen has to be equidistant from the bees at all times to prevent them from getting stressed and leave.

The relationship between a radius and the distance between two midpoints is 1:5.

We create Locus for the queen's movement in GeoGebra. We split the problem into three cases to better understand it. We call the three bees B_1, B_2, and B_3.

Case 1: B_1 circles the periphery while B_2 and B_3 are still.
Case 2: B_1 and B_2 circles their peripheries while B_3 is still.
Case 3: B_1, B_2 and B_3 are all circling their peripheries.

We draw an equilateral triangle with side length (5r). Then we draw three congruent circles with radii (r). The circle centres are placed at the triangle's corners.

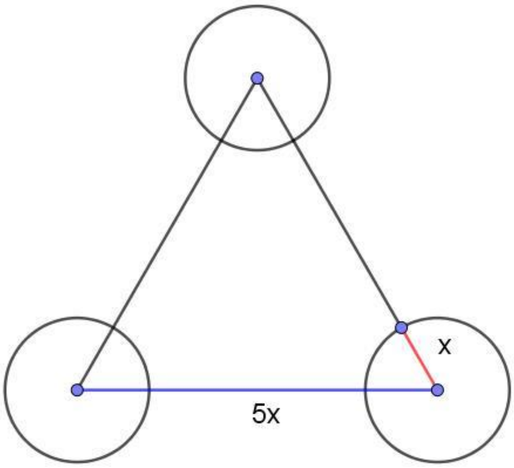


Fig. 12

Case 1

We place a random point on the periphery of each circle. These correspond to the position of B_1, B_2, and B_3. We place the queen, D in a position equidistant from all bee positions. We then move B_1 to a new position along the same periphery. B_2 and B_3 remain stationary. As B_1 changes position, the queen also moves to remain equidistant to all three bees. After several movements, we get a Case 1 Locus for D. Press here

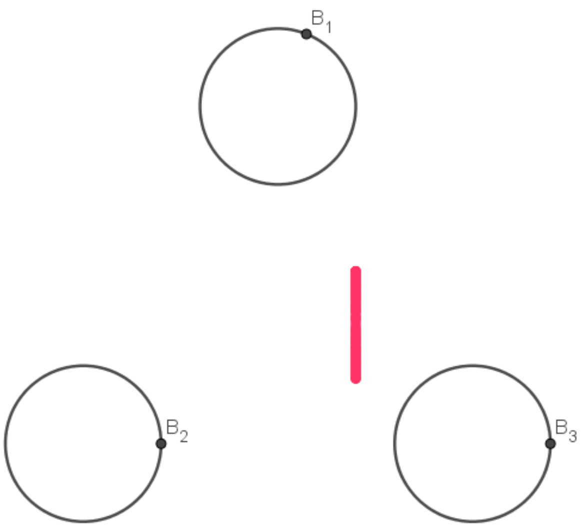


Fig. 13

Case 2

We use the same method as for Case 1 but have both B_1 and B_2 change positions. The Case 2 Locus for D becomes: [Press here]

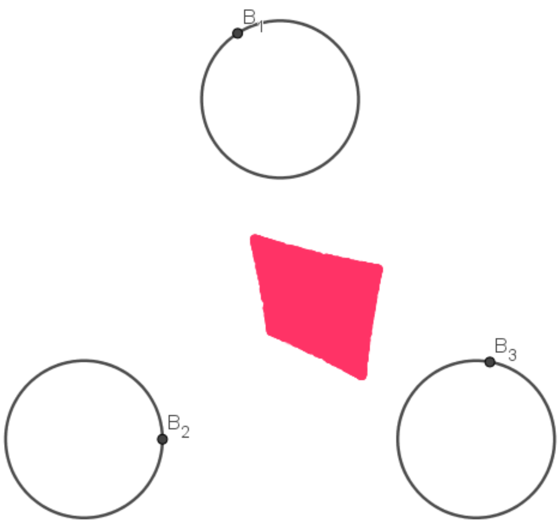


Fig. 14

Case 3

For Case 3, we repeat the process but change all bees' positions. The Locus for D in Case 3 becomes: [Press here]

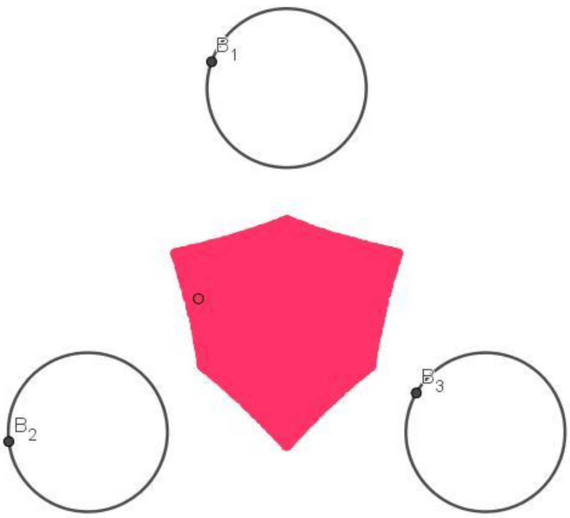


Fig. 15

We have all three Loci overlap (Fig. 16). The first Locus is a line, the second is a Fig. with 4 corners and the last is a Fig. with 6 corners. We study the picture but can't explain why the Loci turn out the way they do. One suggestion is that Loci for Cases 1 and 2 are bordered by hypocycloids, but we can't be certain.

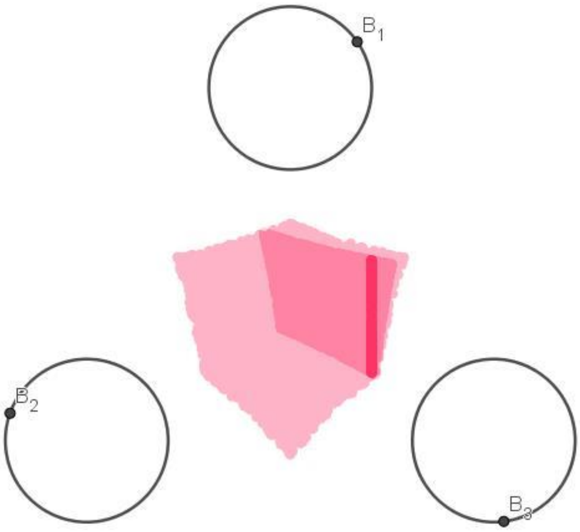


Fig. 16

The number of bees moving determines the shape of the Locus. We see that Locus for Case 3 is symmetrical and assume that this is because the circle midpoints are equidistant.

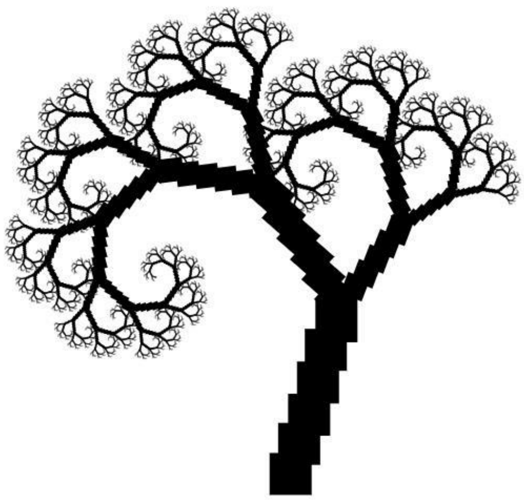
Mandelbrot and Fractals

We chose to write about the mathematician Benoît Mandelbrot (1924-2010). Born in Poland, he moved to France as a child. Mandelbrot is most noted for his work with fractals, as he was the first person to create them digitally.

Fractals are uniform patterns with the same basic shape, repeated endlessly in different scales. The branches of a tree are examples of natural fractals, i.e. fractals found in nature. Each branch has similar, smaller branches. Fractals also have other complex qualities. They have a dimension that is not an integer value, meaning it is impossible to measure a fractal with standard length, area or volume.

Fractals have many uses in today’s society. For example, scientists have used fractals to study lungs and the way they branch, as lungs are structured as natural fractals. Fractals are fascinating and appealing to look at, thus they are often used in art.

To deepen our knowledge of how fractals work and are created, we constructed our own:



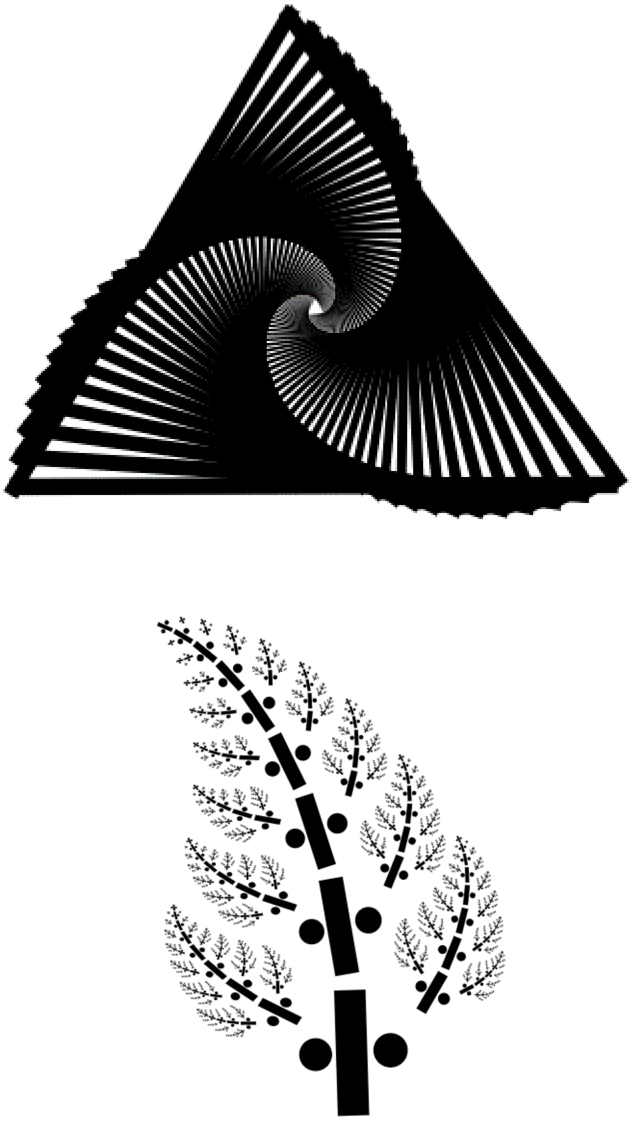


Fig. 17

Before Mandelbrot, Cantor (1845-1918) was one of the first to understand fractals. He created a fractal today known as the Cantor Set, consisting of lines divided into three equal parts. The middle part is repeatedly removed from the lines.

Fig. 18

Another famous fractal is the Koch curve. The sides of the Koch curve start as a line, which divides into three equal parts. The middle part is then removed and replaced by a tip with sides as long as the removed part. The process is infinite.

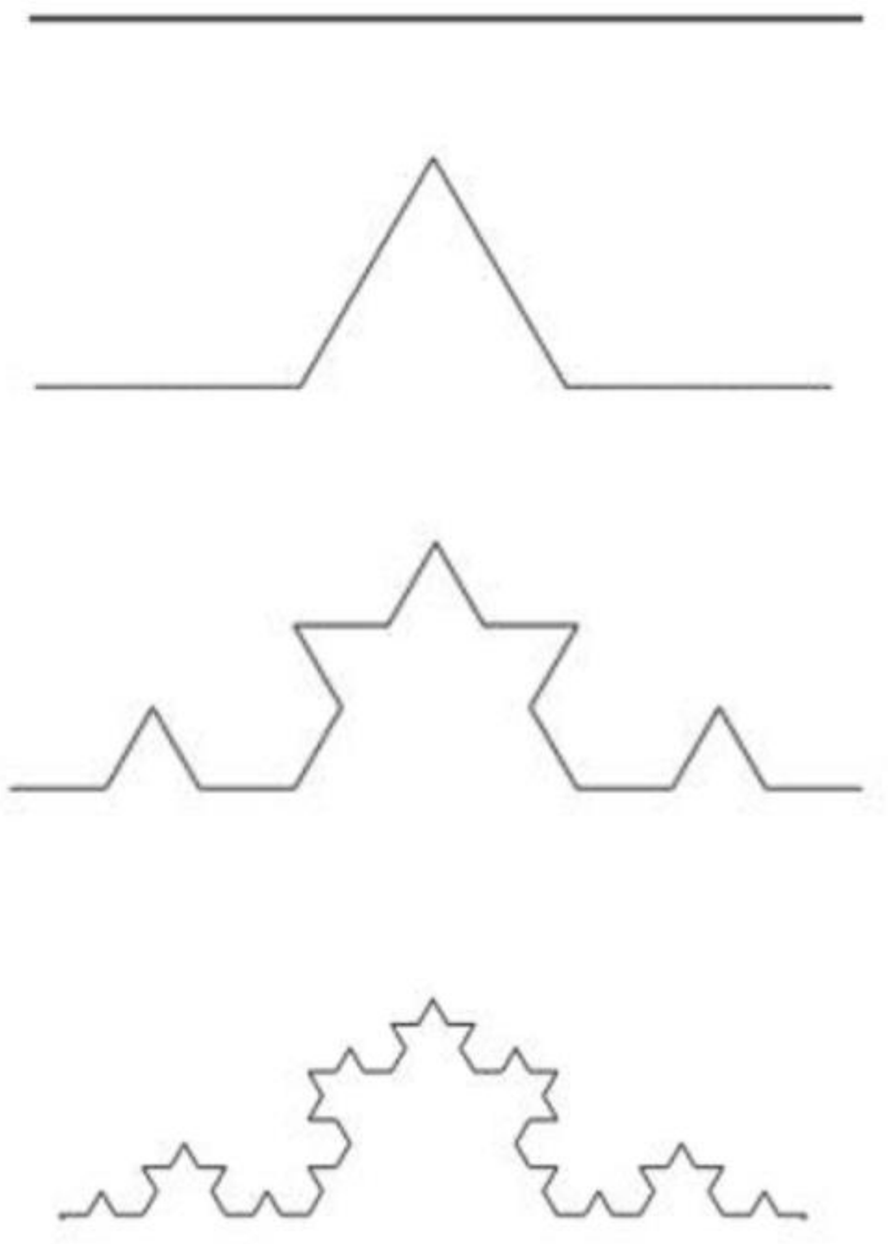


Fig. 19

The Koch curve starts as an equilateral triangle. In the 2^{nd} stage, each side becomes four. In the Koch curve pattern, each Fig. has four times the distances of the previous one.

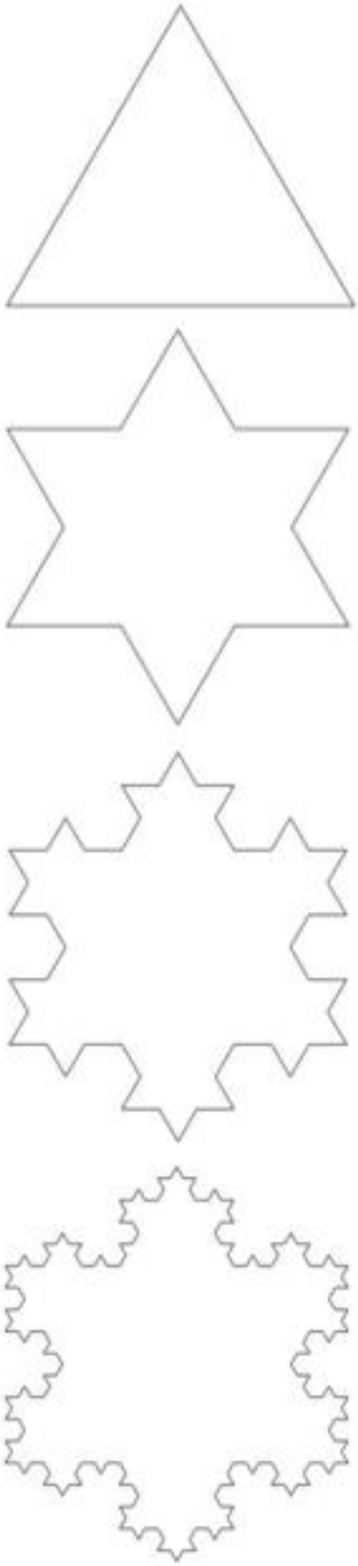


Fig. 20

a_x: $the\ amount\ of\ distances\ in\ figure\ x$

$a_1 = 3 \cdot 4^0$

$a_2 = 3 \cdot 4^1$

$a_3 = 3 \cdot 4^2$

$a_x = 3 \cdot 4^{x-1}$

The length of the distances is $\frac{1}{3}$ of the previous distance.

d_x: $the\ lenght\ of\ one\ distance\ in\ figure\ x$

$d_1 = 1$

$$d_2 = \frac{1}{3}$$

$$d_3 = \frac{1}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^2$$

$$d_x = \left(\frac{1}{3}\right)^{x-1}$$

The total length of the whole curve (L_x) in figure x is the amount of distances multiplied by the length of the distances.

$L_x = a_x \cdot d_x$

$$L_x = 3 \cdot 4^{x-1} \cdot \left(\frac{1}{3}\right)^{x-1} = 3 \cdot \left(\frac{4}{3}\right)^{x-1}$$

Figure 190 calculated in cm:

$$L_{190} = 3 \cdot 4^{189} \cdot \left(\frac{1}{3}\right)^{189} \approx 10^{24}\ cm > 10^{23} cm \approx 10^5\ light\ years$$
$$\approx The\ Milky\ Way\ Galaxy\ extent$$

Summary

We remember when Anna asked us if we wanted to take part in Sigma 8. We all agreed, unaware of the coming late evenings, headaches and conflicts. Despite of the hard work, we laughed a lot and learned much about Locus and geometry, as well as how to complement each other. Cooperation was key. No one can do everything but everyone can do something.

We attended classes remotely when Anna presented our assignment. Split into small groups, we suggested ideas and presented them to the class. After discussing and voting, we had a vague idea on what to do. We split into three groups and worked on the separate parts.

The groups tried to solve the first assignment in different ways. Drawing by hand, using GeoGebra or just trying various ideas. We realized that the Pythagorean Theorem in GeoGebra is the most efficient option for Locus problems.

As we designed our new Locus cases, we all looked for inspiration and ideas online. Thanks to our aggregated knowledge of Russian, Persian, Mandarin and Japanese, we found lots of information.

The Part 2 group held a vote to choose three problems to work on, split into smaller groups. The bee group contacted an external math teacher, Jonas Hall. He helped us develop and structure the problem by showing us unfamiliar functions and tools in GeoGebra.

Locus requires focus, which would have been impossible to maintain without Anna providing us with food, Swedish fika and breaks. We are also eternally grateful for the access to principal Karin’s credit card.

Pictures

Sources

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- https://pl.wikipedia.org/wiki/Spirograf_(przyrz%C4%85d)
- http://recursivedrawing.com/
- https://www.wolframalpha.com/

External persons:

- Jonas Hall
- Eeva Klintfors
- Stina Hallén

Pictures:

- https://www.pinterest.se/pin/114124036179309 0909/
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