NMCC 2022

THE FASCINATING WORLD OF TETRAHEDRA

Olari School 8F

Finland

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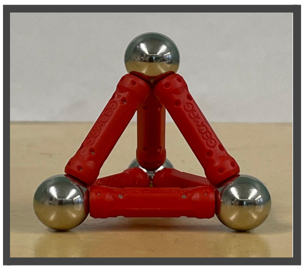
1. Introduction

In 2022 our class was very honoured to participate in the NMCC-mathematics competition. During the NMCC project-investigation, we were to inspect different sizes of tetrahedra and find formulas regarding the amount of rods and spheres used for each size. On top of that, we explored and studied various fractals.

During the investigation, we built lots of tetrahedron models using GeoMag-supplies and planned an exhibition on fractals. We pondered further on the topic and each student learned a lot about problem solving and the basics of solid geometry. In this report we are very excited to introduce you to our findings.



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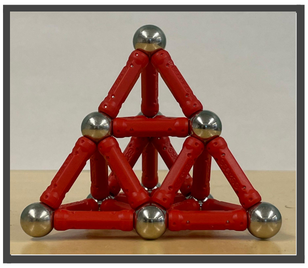

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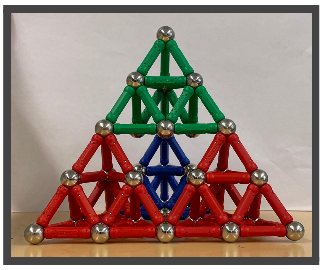

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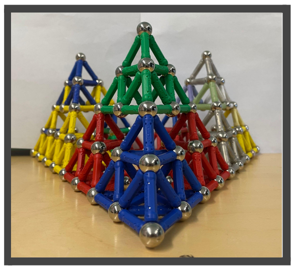
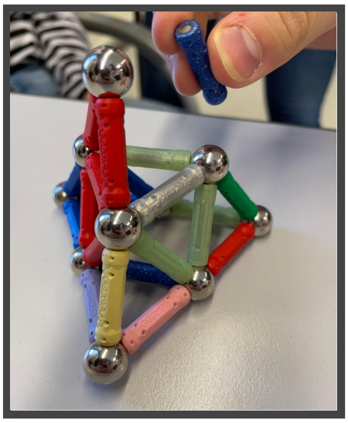
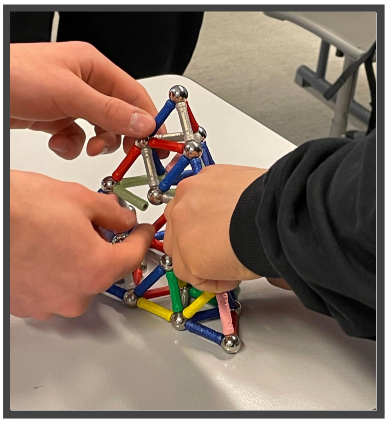


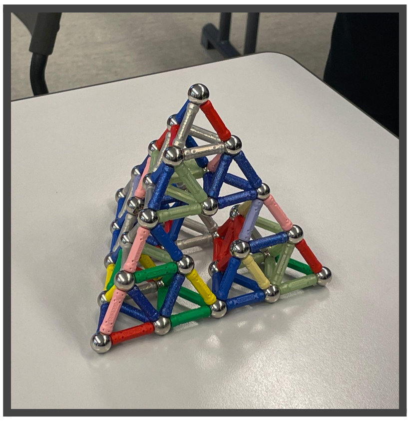
Size 4. (base)

2. Investigation

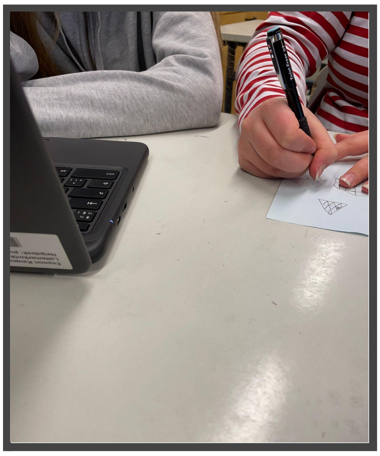
During the investigation, students got orientated with tetrahedra and fractals by building models of them. They shared their own ideas and views, and helped each other. Unfortunately our teacher was sick most of the investigation-period, so students were required to take the initiative. This report is therefore produced by students alone without any guidance from teachers.

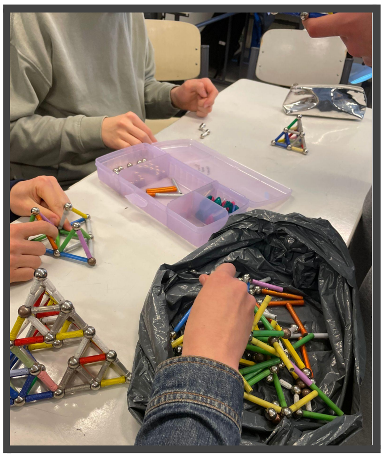






28.3.2022: Pupils were introduced to the task ahead.





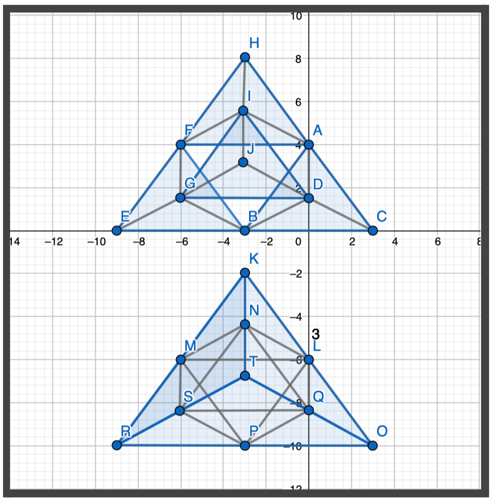
5.4.2022: Pupils split into groups based on their strengths.

3. Report

In this assignment, we explored the connections between the amount of rods and spheres required to build a tetrahedron. Since tetrahedra grew regularly, we acknowledged that there must be a pattern for the growth. In the next paragraphs, we will introduce you to our solutions for these questions.

Size	1	2	3	…	n
Rods	6	24	96	…	x
Spheres	4	10	34	…	y

3.1 Tetrahedron as a concept



A tetrahedron is a quadrilateral (a certain type of polyhedron) with four triangle-shaped faces. Tetrahedra are three-dimensional simplexes, where one-dimensional refers to a segment and two-dimensional a triangle. In this case, we are inspecting regular tetrahedra, which are quadrilaterals with equal length edges, where all faces are shaped as equilateral triangles.

There are two tetrahedra in the picture. The upper image shows four tetrahedra – one edge from point A to point C, put into the shape of a tetrahedron of the next size. Underneath, it displays a large tetrahedron that can be formed from an infinite amount of smaller ones. Objects with a feature that can repeat itself infinitely are called fractals.

Source: GeoGebra, Anja Miao

3.2 Rods required

The amount of rods used in different sizes of tetrahedra form a sequence. Sequences can be arithmetic or geometric, depending on if the difference or ratio is common between consecutive terms. For rods required, we observed a geometric sequence. Knowing that the next size forms from four tetrahedra of the previous size, the amount of rods required for the next size is therefore always the previous size value multiplied by 4, leading to the ratio between terms being four.

Size	1	2	3	…	n
Rods r_n	6	24	96	…	x
Solution	6	$6 \cdot 4^1$	$6 \cdot 4^2$	…	$6 \cdot 4^{n-1}$

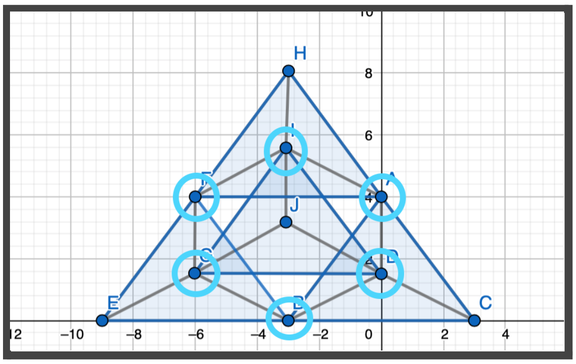
Solution: $r_n = 6 \cdot 4^{n-1}$, n = size, 6 = rods in size 1, 4 = ratio between neighbour values.

3.3 Spheres required

The next problem was to determine how many spheres are required for each size tetrahedron. In the beginning of solving this, we built many models from GeoMag-supplies and examined the change inside the structure. This task was significantly harder, but after extensive hard-work and exploration, a pupil came up with a solution.

Initially, we wondered if multiplying the previous value by four, like in the former task, we could find the amount of spheres required for the next size. This hypothesis turned out wrong as shown in the table below:

Size	1	2	3	4	5
Spheres s_n	4	10	34	130	514
idea 1: $s_n = 4s_{n-1}$	4	$4 \cdot 4 = 16$	$4 \cdot 16 = 64$	$4 \cdot 64 = 256$	$4 \cdot 256 = 1024$

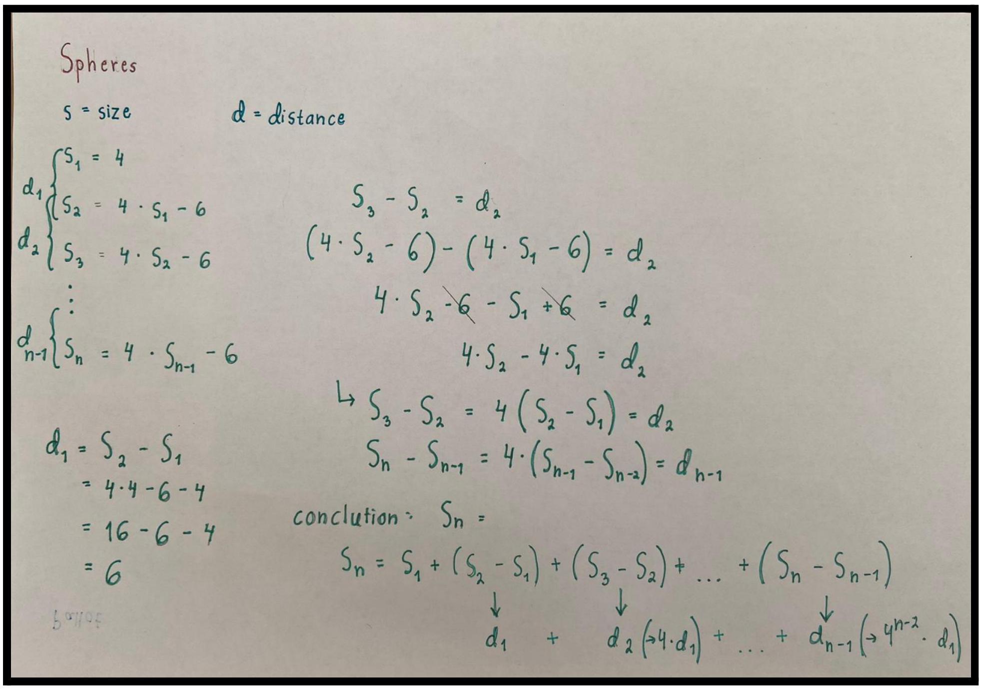


Although this theory didn't work, we realised that subtracting a certain amount of spheres s_n from the amount calculated with $4 \cdot s_{n-1}$, which are the spheres shared in the six junctions, we can solve the amount of spheres required. With this piece of information, we came up with a formula for spheres required s_n in the next size: $4 \cdot s_{n-1} - 6$.

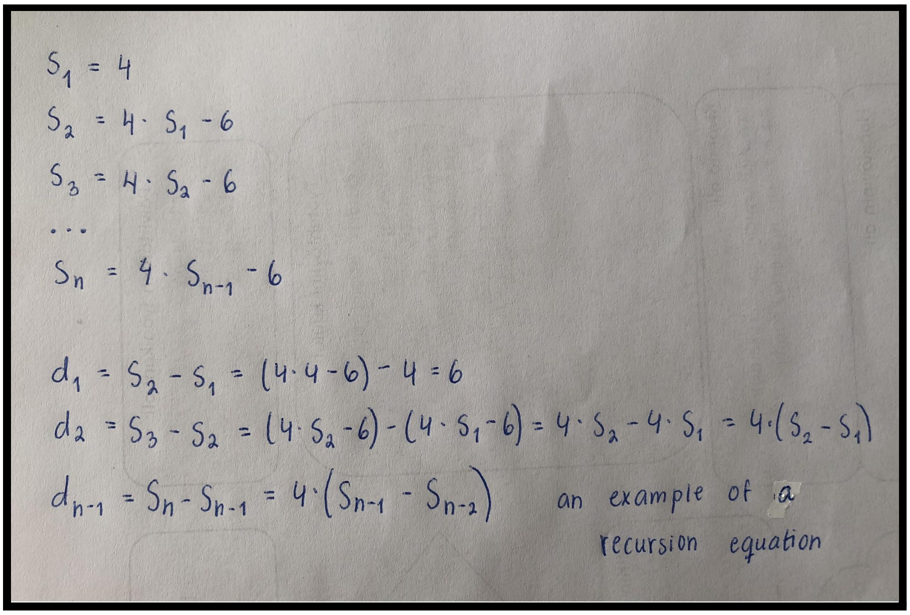
Size	1	2	3	4	5
Spheres s_n	4	10	34	130	514
idea 2: $s_n = 4s_{n-1} - 6$	**4**	$4 \cdot 4 - 6 =$ **10**	$4 \cdot 10 - 6 =$ **34**	$4 \cdot 34 - 6 =$ **130**	$4 \cdot 130 - 6 =$ **514**

Although we had successfully found a solution to solve the amount of spheres required for the next size, this solution was only practical for calculating smaller and consecutive sizes. Continuing from here was not easy, but the class worked diligently, driven by motivation and faith in each other. Since the sequence formed from the values of spheres required was neither arithmetic nor geometric, this task was notably harder.

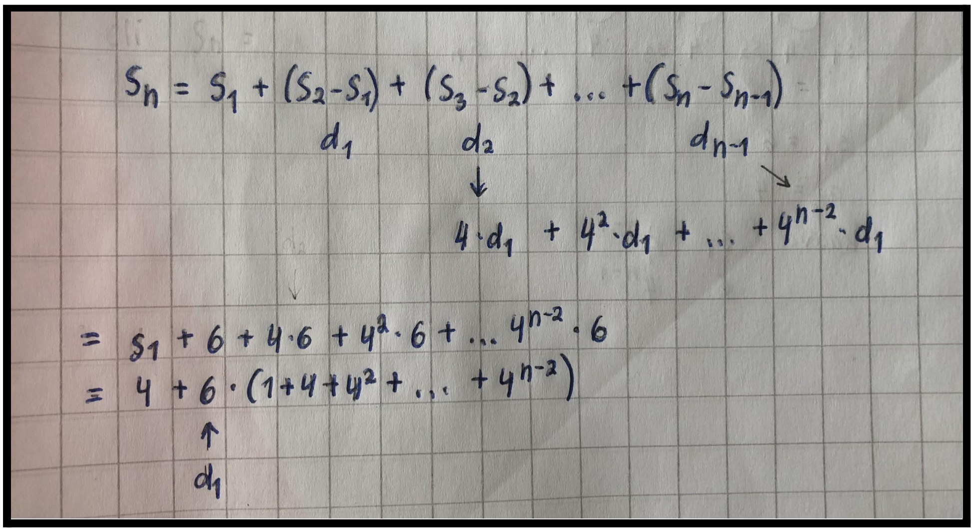
Solution 1:



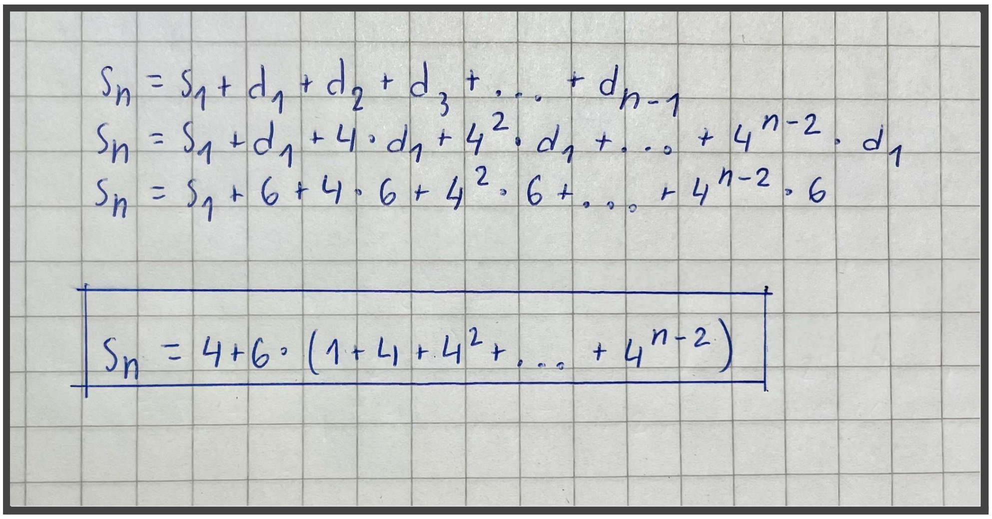
After working persistently, one pupil came up with a solution, when she investigated the distance between each consecutive term. In the picture above, the values we know beforehand have been written down. We know, that in size 1 there are 4 spheres, leading to there being $s_2 = 4 \cdot s_1 - 6 = 4 \cdot 4 - 6 = 10$ spheres in size 2, and $s_3 = 4 \cdot s_2 - 6$ in size 3, etc. Since the sequence doesn't follow the rule of an arithmetic nor geometric sequence (due to the subtract of 6 balls in each recursion step), we have to examine the distance between the amount of spheres required in each adjacent size. The distances are named with the letter *d*. Below we demonstrate the difference of spheres required in two adjacent sizes. With a simple recursion equation, we can calculate the wanted function using the values known beforehand.



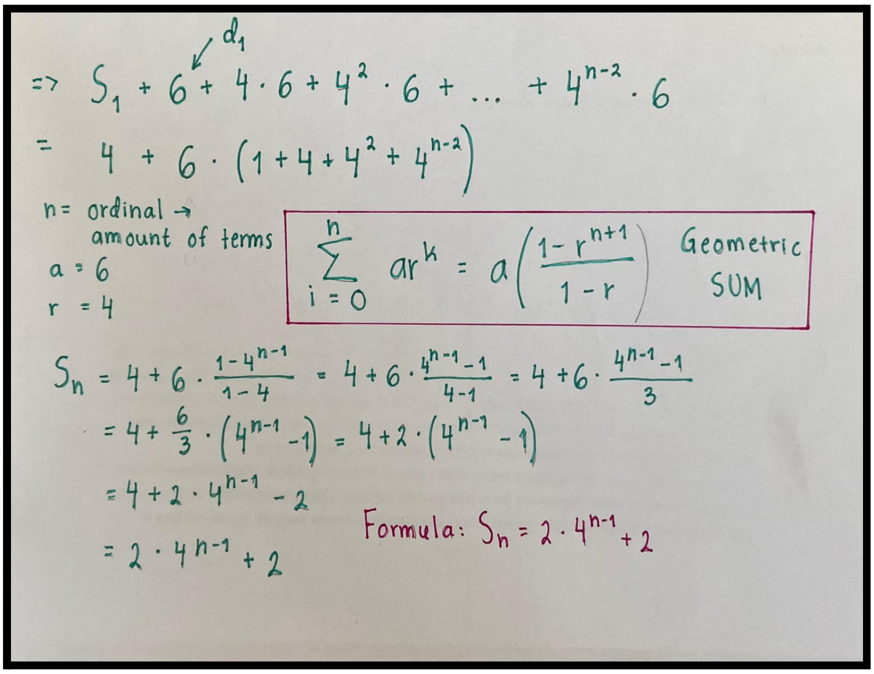
With these perceptions we can solve the amount of spheres in size n. This works because we know the connection between any contiguous sizes.



By adding all the differences of two adjacent sizes, i.e., the distances up to size n, to the spheres required in size 1 we can calculate the amount of spheres required in size n.



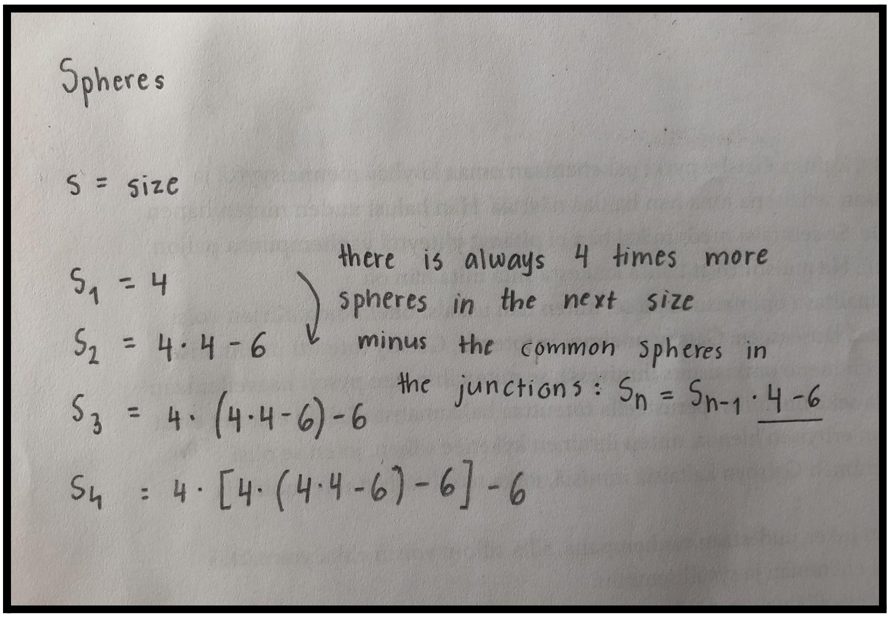
In the simplified version, we are working on a calculation with strong features of a geometric sequence. Now, we've found a wanted formula that works on any given size of a tetrahedron. The formula could be left into the form: $s_n = 4 + 6 \cdot (1 + 4 + 4^2 + \ldots + 4^{n-2})$, but it can still be simplified to be more practical with a simple mathematical trick.



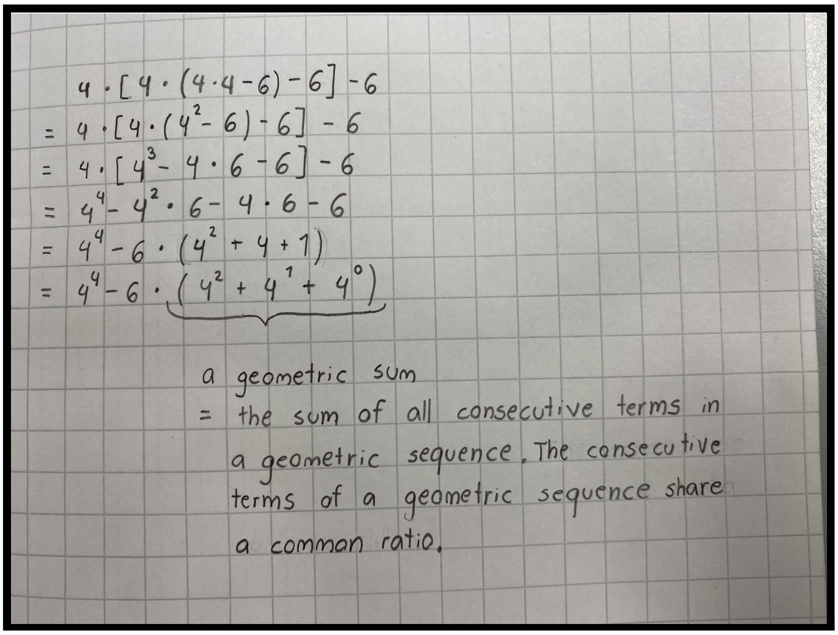
Solution: $s_n = 2 \cdot 4^{n-1} + 2$

Although the class had obtained a formula, we were determined to find another one. Continuing from here was tricky, since a large part of the class was stuck with the previous solution or lost on their own path to enlightenment. When one pupil started thinking from the very beginning, she was able to look at the task from a new perspective.

Solution 2:



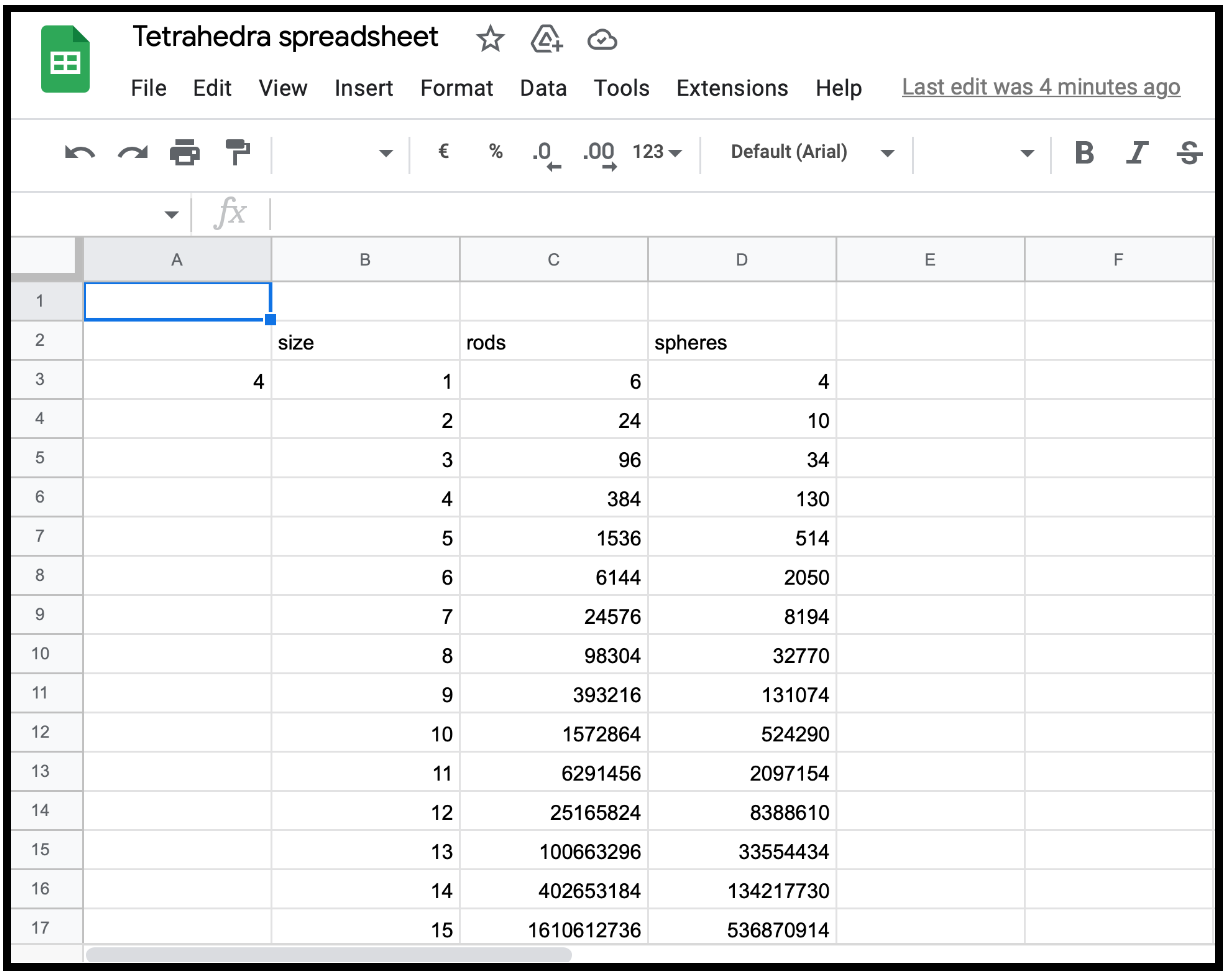
It is known that the amount for spheres in continuous sizes is always 4 times the amount of spheres in the previous size minus the amount of spheres shared in each junction. Before, we had never continued calculating from here, which led to numbers 4 and 6 dancing around in the formula, and the formula increasing in length as the amount of terms increased. As the result, this pupil simplified the formula for s_4 in this way:



Here, we notice that $4^2 + 4^1 + 4^0$ forms a geometrical sum because the ratio between each consecutive term is common. Knowing $4^4 - 6 \cdot (4^2 + 4^1 + 4^0)$, we can form another unique formula for the spheres required in the size n.

Solution: $s_n = 4^n - 6 \cdot (4^{n-2} + 4^{n-3} + 4^{n-4} + \ldots + 4^0)$

During our research we came up with a few more solutions. In addition to the formulas introduced above, one pupil came up with a working solution whereby adding the amount of spheres s_{n-1} and rods r_{n-1} used in the previous size, you'll get the amount of spheres s_n used in the next consecutive size: $s_n = s_{n-1} + r_{n-1}$.



	A	B	C	D	E	F
1						
2		size	rods	spheres		
3	4	1	6	4		
4		2	24	10		
5		3	96	34		
6		4	384	130		
7		5	1536	514		
8		6	6144	2050		
9		7	24576	8194		
10		8	98304	32770		
11		9	393216	131074		
12		10	1572864	524290		
13		11	6291456	2097154		
14		12	25165824	8388610		
15		13	100663296	33554434		
16		14	402653184	134217730		
17		15	1610612736	536870914		

The solution above can be proved using solution 1:

$$s_n = (s_{n-1}) + (r_{n-1}) = (2 \cdot 4^{n-2} + 2) + (6 \cdot 4^{n-2}) = 8 \cdot 4^{n-2} + 2 = 2 \cdot 4^{n-1} + 2.$$

When we compare the obtained solutions, we notice that in terms of mathematics, solutions 1 and 2 are better because the amount of the spheres can be determined without knowing the amount of previous size's spheres. Solution 1 employs a special technique to expose the nature of the derived geometry sequence, and the procedure is consequently less straightforward. Solution 2 achieves the formula by directly unfolding the expression and regrouping the terms in a smart way.

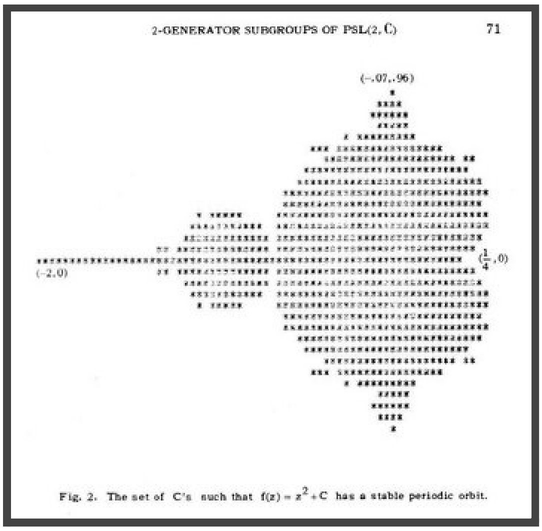
Our class' general opinion is that formula 2 is easier to understand, while formula 1 is more compact for larger sizes. Both solutions are equivalent. The theory mentioned above whereby adding the amount of rods and spheres required in the previous size we can easily figure out the amount of spheres in the next size, and it's also very useful and simple to understand, when knowing the previous values. The latter is also easy to teach and very intriguing when compared to how hard the task was initially.

3.4 Fractals

Fractals are shapes that contain certain special features. They consist of similar or identical characteristics that can duplicate themselves infinitely, so that when inspected from any magnification, the feature doesn't change. As mentioned, tetrahedra are examples of fractals.

We have all consciously and unconsciously seen fractals. They can be seen around us on a daily basis, in nature, the human body, art and architecture and elsewhere – e.g. every tree is a fractal of its own kind because the branches branch continuously into more identical ones. Other common natural fractals are mountain silhouettes, lightning bolts and snowflakes. Even the air passages in lungs split into smaller duplicates of the previous one. Fractals can also be noticed in ancient drawings.

Fractals were first discovered in the 19th Century. However, proper research was only launched shortly after efficient computers were invented. When one of the most well known fractals, the Mandelbrot set, was first generated with computers, the researchers believed they had made a mistake, since the result was so unexpected.



In the picture, we can see the Mandelbrot set generated with a computer first created by Robert Brooks and Peter Matelski in 1979. The fractal is named after the inventor, a French mathematician named Benoit Mandelbrot, who first visualised it with a computer in the 1960s. Source

Fractal geometric concepts have been a part of mathematicians' and programmers' everyday life since the mid-1980s.

4. Summary

We started working on this assignment by dividing into smaller groups. At first, everyone had trouble grasping the new terms, but we tried our best. During each lesson, our class had various heated conversations about the topic that were far from useless, because they opened our eyes for new possibilities and helped us see and create new solutions. For some it was quite challenging understanding new concepts, but for others it was more simple. Overall, it was a very enlightening process.

During this project everyone learned a lot, especially about geometry and problem solving, and as time passed, we were able to utilise knowledge learned before and e.g. link the Fibonacci-sequence to some fractals. This project also gave us an opportunity to practise teamwork skills and raised our class spirit. May this stand as a reminder that mathematics isn't just calculations done by yourself, but more importantly discussing and wondering together, for it can give you infinite opportunities to discover newly found and unique solutions.

During this investigation all of us were able to participate and learn, so thank you for reading our report and giving us this opportunity to learn and discover!

5. Sources and applications used

Tekniikan Maailma. Fraktaalit osoittavat, kuinka kaunista matematiikka voi olla Read 5.4.2022

Matemaattinen blogspot. Fraktaaleista Read 5.4.2022

Goaravetisya. Fraktaali- ja fraktaaligeometrian käsitteet. Read 11.4.2022

Wikipedia, article: tetrahedron. Tetraedri – Wikipedia Read 12.4.2022.

Matemaattinen blogspot. Mandelbrotin joukon generointi tietokoneella Read 17.4.2022.

Jari Mäkinen. Tiedetuubi. Päivän kuva 11.5.2013: Matematiikka on kaunista | Tiedetuubi Read 17.4.2022.

Wikipedia, article: recursion. Rekursio – Wikipedia Read 18.4.2022.

GeoGebra Classic

Google Docs

Google Sheets

Papers

GeoMag-magnetic rods and spheres

All pictures, except the Mandelbrot-set, were made by students