Tetrahedra

NMCC

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Summary

For a couple of weeks, our class has been working on the task of tetrahedrons. The goal has been to come up with a formula that describes the number of sticks and balls in a desired figure. It was a difficult task, but we worked hard together to come up with a solution. This report describes in detail what we did in class, how we worked, what and how the resources were utilized. We have stated the problems we encountered and how we solved them, the conclusions drawn from the results and learnings during the work process.

Conceptual Knowledge

The process initiated by learning and understanding the current concepts, which were *tetrahedrons* and *fractals*.

A tetrahedron is a polyhedron which has four corners and six edges, consisting of four triangles with three side surfaces that meet in each corner.

The figure we worked with was a regular tetrahedron, which means that its four sides are equilateral triangles.

Figure 1 Geomag.

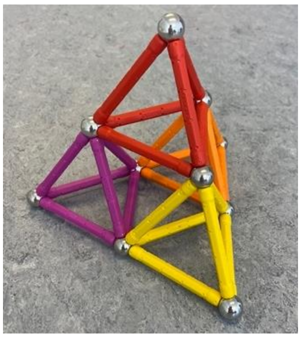


Figure 2 Geomag.

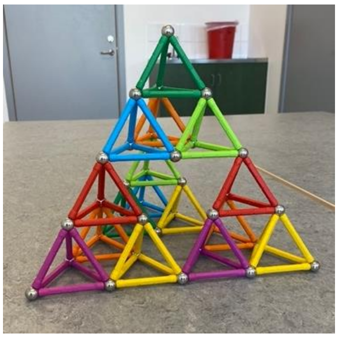
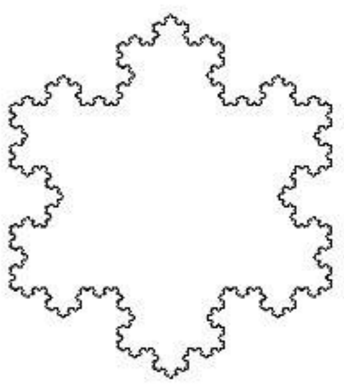


Figure 3 Geomag.

Fractals is another subject matter we explored in depth. They are geometric patterns that have the same structure regardless of scale. This means the pattern repeats itself infinitely. The pattern can be both two- and three-dimensional and can be found in nature. A well-known example of a natural fractal is the plant Broccolo. There are also manmade fractals; Von Koch's snowflake is one of the examples, and it has a limited area like a regular square but still has an infinite circumference.

The plant Broccolo is.a fractal.



Von Koch's snowflake has an infinite circumference

Interpretation

We interpreted the task to find a separate formula for the number of sticks and balls used in the figure n. We had recently finished working with Algebra in class which helped us understand that when it comes to a pattern problem, the first focus should be on shaping a formula.

Work and Resources

As a class, we had one lesson every week in which we worked in small groups solving the problem tetrahedron. We made small groups to ensure that everyone in each group got the opportunity to work on the task and feel secure, confident, and contributing. This way we got the ideas from different perspectives and angles. We adopted this method to develop teamwork spirit and to diminish the complete reliability on one or few students in the group.

In the very first lesson, we tried to make models of the tetrahedron with cotton balls and wooden sticks to understand the problem better. It turned out to be a difficult and time-consuming task. Meanwhile, some of us drew 2D models of the figures which did not work very well either.

We, therefore, decided to find another solution, which led us to use “Geomag”, which was in fact both easier and faster. During the following lessons, we used “Geomag” to build the figures and came up with a solution faster. The reason we were building up the models was to get a better understanding and a holistic picture of how the figures were structured.
We thought our idea of getting a better understanding of the structure would help us find a solution, which turned out to be true.

When we built the models, it became easier for us to understand the connections and patterns. Some of the students made charts with the number of sticks and balls in the first few figures we built. This facilitated to split the tasks among the classmates. A few students started to analyze the charts and tried to find out the formulas while others continued to analyze the models. There was also a group checking the different solutions coming from everyone. This proved to be a fantastic method with a good variation of solutions coming from different groups, depending on if one was working with charts or models of the figure.

Despite working in small groups, we had to share the “Geomag” as we did not have enough. The groups were hence mixed, and the calculations made by previous groups spread quickly in the new groups. Although the class worked together, everyone did not have the chance to understand exactly how the solutions worked. By mixing the groups everyone got the chance to talk and listen. As a result, we got an even wider range of solutions.

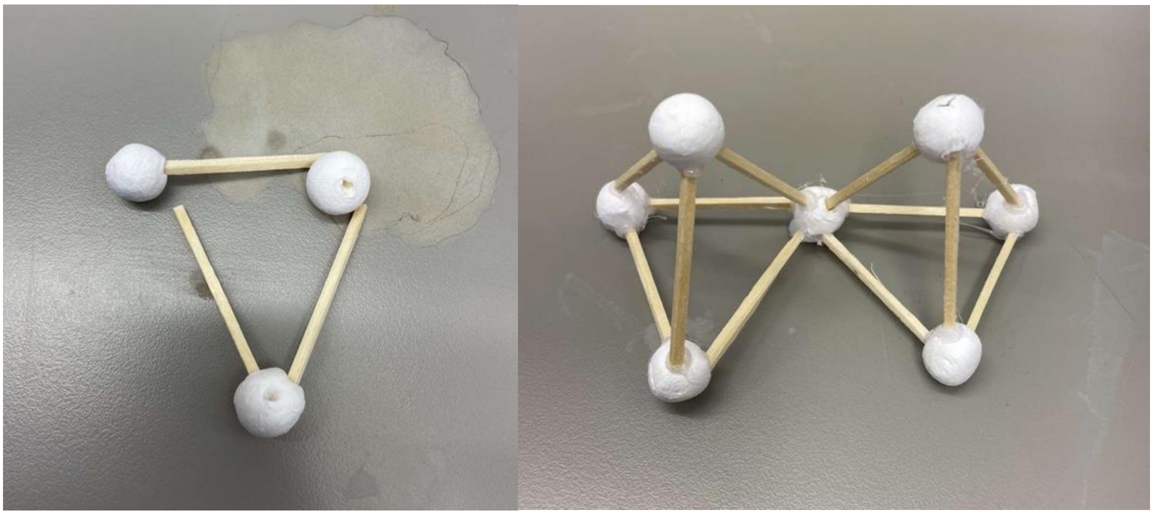
In the lesson before the groups were mixed, we had already reached a general formula for the number of balls and another for the number of sticks. Feeling this was the goal of the task we all sat down to review and summarize all the results we had derived. We discussed the advantages and disadvantages of the different solutions and concluded that the general formulas were the best.

Problems

During the entire work process, we encountered a couple of problems. In the beginning, when trying to build the models, we only had cotton balls and wooden sticks to use. We tried to push holes in the cotton balls to attach the sticks, but it was difficult to build since the model was breaking. To glue the sticks together worked better but still was not an effective method. We learnt from our mistakes and brought our own "Geomag" to the school for the next lesson to build models. The plan with "Geomag" worked amazingly well, even though we didn't get much work done the first lesson.

We had a bit of a problem in mathematics to come up with a general formula for the number of balls in the figure. On the contrary, finding the solution for the sticks was rather immediate. Since the balls were divided among the smaller tetrahedra in the tetrahedron, we could not use the same method to solve the task for the balls as we had used for the sticks.

However, we took a prompt action when we realized the problem. We began to interact and co-operate among different groups. With these discussions, many new and important perspectives were raised, which in turn ignited further ideas. That enabled us to move forward in the right direction and finally arrive to a solution despite the initial setbacks.



Pictures of our models from the first lesson

Solutions:

We derived some key results that we think are important and would like to share in the report.

- A method of calculating:
  - The amount of balls in a figure by knowing the amount of sticks and balls in the previous figure.
  - The amount of sticks in a figure by only knowing the amount of sticks in the previous figure.
  - The amount of balls in a figure by only knowing the amount of balls in the previous figure.
- Our Goal – To come up with a general formula for:
  - The amount of sticks in figure n.
  - The amount of balls in figure n.

We are grateful that we all think in different ways and from different perspectives. If it had not been for some of us drawing tables to analyze, we would have missed this. As seen in Table 1, the number of balls in a desired figure equals the sum of the balls and sticks in the previous figure.

Table 1

Figure (n)	Sticks (a)	Balls (b)	$a_n + b_n = b_{n+1}$
1	6	4	10
2	24	10	34
3	96	34	130
4	384	130	514

<u>Note that a "box" in the text is the same as a figure 1 in the task, i.e., a tetrahedron consisting of six sticks and four balls.</u>

We found a simple way to calculate the number of sticks in a figure if you only know the number of sticks in the previous figure. You simply multiply the number of sticks in the previous figure by four and you will get the number of sticks in the desired figure. This is because the new figure consists of four times the number of boxes than the previous figure.

Table 2

Figure (n)	Sticks (a)	$a_n \cdot 4 = a_{n+1}$
1	6	24
2	24	96
3	96	384
4	384	1536

To calculate the number of balls in a figure, we thought of using the same method as for calculating the number of sticks. It did not come out accurate, since some of the balls were divided between boxes. We could see that six balls acted as joints to connect the four replicas of the previous figure at different points. These balls connected two of the previous figures each. Therefore, when the number of balls in the previous figure is multiplied by four, one has to subtract the six balls that were counted twice in the multiplication.

When we encountered hiccups while coming up with a general formula for the number of balls, we found this property of the tetrahedron, and we were confident we were on the right track.

Chart 3

Figure (n)	Number of balls (b)	$b_n \cdot 4 - 6 = b_{n+1}$
1	4	10
2	10	34
3	34	130
4	130	514

The number of sticks is a simple geometric sequence which can be described as:

$$a_n = a_1 \cdot k^{n-1}$$

We realized that it was a geometrical sequence because the number of sticks multiplies by 4 for each new figure which means the quota, or *k*, also equals 4. Figure 1 has 6 sticks which means $a_1 = 6$. The formula we then came up with looked like this:

$$6 \cdot 4^{n-1}$$

This formula was later simplified like following:

$$6 \cdot 4^{n-1} = 1{,}5 \cdot 4 \cdot 4^{n-1} = 1{,}5 \cdot 4^n$$

Our finished formula to describe the amount of sticks in figure n:

$$1{,}5 \cdot 4^n$$

Table 4

Figure (n)	Number of sticks (a)	$k = \frac{a_{n+1}}{a_n}$	$a_n = a_1 \cdot k^{n-1}$
1	6	$\frac{24}{6} = 4$	$6 \cdot 4^0$
2	24	$\frac{96}{24} = 4$	$6 \cdot 4^1$
3	96	$\frac{384}{96} = 4$	$6 \cdot 4^2$
4	384	$\frac{1536}{384} = 4$	$6 \cdot 4^3$
n	$6 \cdot 4^{n-1}$	$\frac{a_{n+1}}{a_n} = 4$	$6 \cdot 4^{n-1}$

As we mentioned earlier in the report, it was more difficult to come up with a general formula for the number of balls as they are divided between boxes. Nevertheless, we managed to come up with a formula that works for all figures, and it looked like this to start with:

$$\frac{4 \cdot 4^{n-1} - 4}{2} + 4$$

First, we had to come up with a way to describe the number of boxes in a figure. Since no sticks are divided between boxes, the number of boxes can be described as the amount of sticks divided by the number of sticks per box. The number of boxes in a figure is therefore equal to:

$$\frac{6 \cdot 4^{n-1}}{6} = 4^{n-1}$$

We also understood that all balls are divided between two boxes except the four corner balls.

1. Since each box has four balls, we took the number of boxes (4^{n-1}) multiplied by the number of balls in each box (4).

$$4 \cdot 4^{n-1}$$

2. Subtract the corner balls to get the amount of balls, that are not corner balls, counted twice.

$$4 \cdot 4^{n-1} - 4$$

3. Divide it by two and add the four corner balls again. This equals the number of balls in figure n.

$$\frac{4 \cdot 4^{n-1} - 4}{2} + 4$$

4. Our formula was then simplified.

$$\frac{4^1 \cdot 4^{n-1} - 4}{2} + 4 = \frac{4^{n-1+1} - 4}{2} + 4 = \frac{4^n}{2} - \frac{4}{2} + 4 =$$

$$= \frac{4^n}{2} + 2$$

Reflection and Learnings

The entire work process was very educational and intuitive for the class. The tasks were challenging and made us work hard together as a team. It was a great opportunity to train our abilities and use our skills to solve the problems during different situations. Even for those who easily understood the tasks and came up with solution ideas, had to train their abilities to teach and explain applied mathematics to others in a simple way. We developed our techniques to work with patterns and learned to interpret the correlations between figures.

To share ideas, brainstorming sessions and taking help from each other to solve the tasks you are not comfortable with, was most challenging yet exciting. We felt we got stuck at times with nowhere close to the solution. With support from classmates and thinking from new perspectives we managed to achieve the solutions which we all are very happy and satisfied with.

List of Sources

1. NE.se
   1.1. https://www.ne.se/uppslagsverk/encyklopedi/l%C3%A5ng/fraktal
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