ELLIPSES

Nordic Math Competition

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8.B

Denmark
30/05/23

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INTRODUCTION

We have worked with ellipses by using different methods. We have chosen to investigate the ellipse and its properties with different methods such as conic sections, paper folds and nails and string. We would like to investigate the properties of the ellipse, including the focal points and eccentricity of the ellipse. We will do this through our previously mentioned methods.

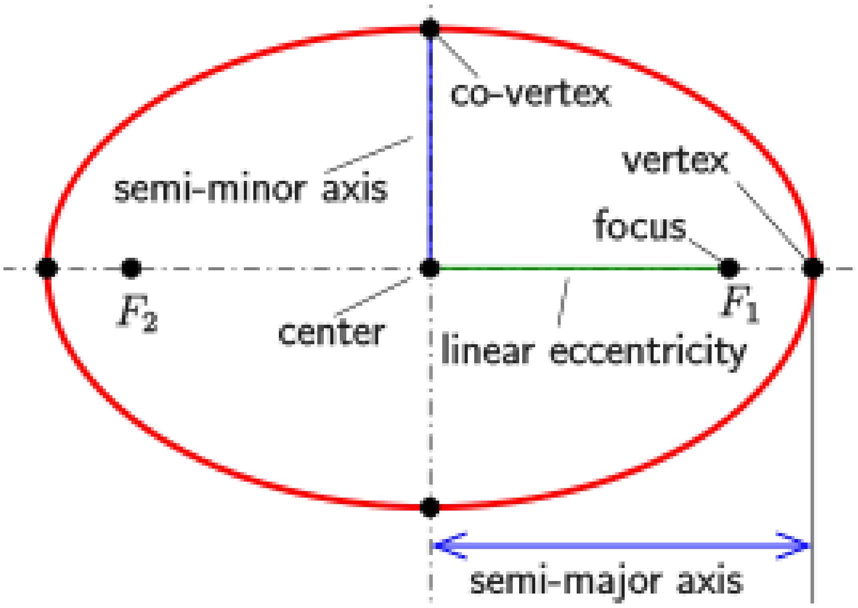
What is an ellipse?

The definition of an ellipse is all points in a Cartesian coordinate system where the sum of the distance between two arbitrary focal points(foci) F_1 and F_2 is a constant.

An ellipse consists of different parts. Here are a few of them: The main ellipse, the foci, the focal rays: a line from a focus to an arbitrary point on the ellipse, the minor axis and major axis.

Foci and Focal rays

Foci: The focal points of an ellipse are the two center points. Which in an ellipse contains two points and in a circle one point.



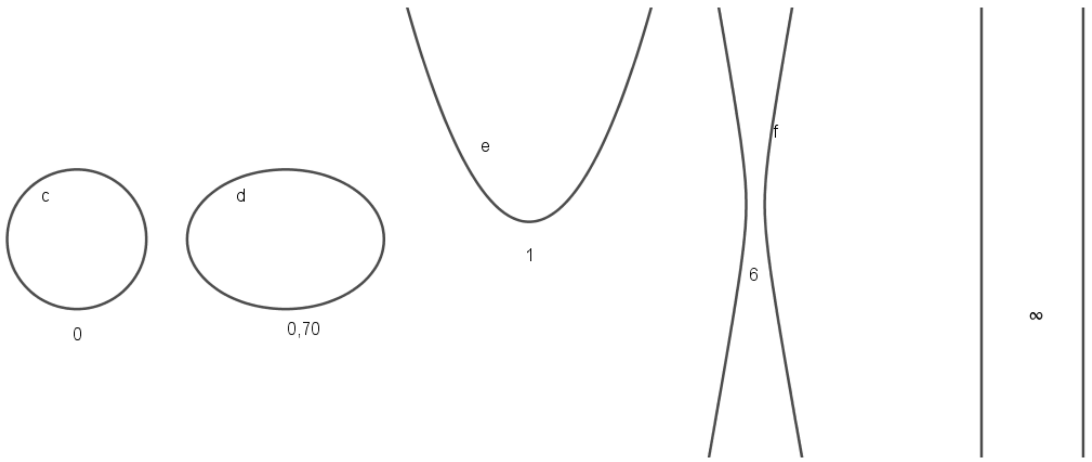
Focal rays: The focal rays are the two lines which go from the Focal points to a point on the ellipse. (Where the point on the ellipse is, doesn't matter). No matter where the point it is, the sum of the two focal rays will always be the same as the length of the major axis.

Minor and major axis

Minor axis: The minor axis is the line which goes along the smallest diagonal of the ellipse and exactly in between the focal points. It is perpendicular to the Major axis.
Major axis: The major axis is the line which goes along the ellipses largest diagonal and ends at the two top points

Eccentricity

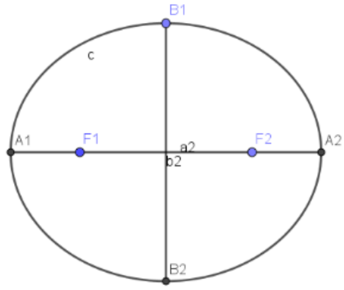
The eccentricity is the shape of the ellipse. You could also think of it as how squished the ellipse is. The bigger the eccentricity is, the more squished it is. If the eccentricity is between 0 and 1 the shape is an ellipse. But if it hits 1 it becomes a parable and if it exceeds 1 it's a hyperbola. If the eccentricity is infinite, it's two straight lines. All examples can be seen below.



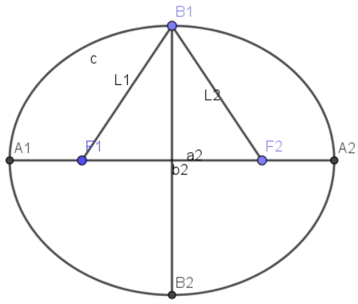
$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

This is the standard equation for the eccentricity of an ellipse, but how can you derive it from the definition of the eccentricity of an ellipse:
The eccentricity of an ellipse is defined as so:

$$e = \frac{|F_1 F_2|}{|A_1 A_2|}$$

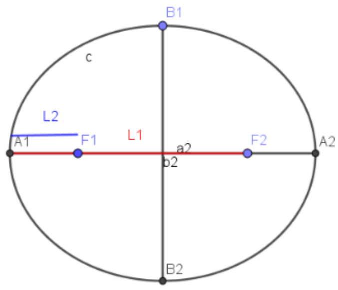


By making a triangle $(\Delta F_1 F_2 B_1)$ we can use that triangle in our proof, but first we must find out what the triangle is. We know that B1 is on the ellipse and f1 and f2 are the foci of the ellipse so the sum of the lines b1f1 and b1f2 together is a constant K is the constant but it can be expressed as something else. By making the construction shown we can derive that by moving L2 the constant is the same as the major ax $(|B_1F_1| + |B_1F_2| = K)$



$$|A1F1| + |A1F2| = |A1F1| + |A2F2| + |F1F2|$$

$$|\mathrm{A1F1}| + |\mathrm{A2F2}| + |\mathrm{F1F2}| = 2\mathrm{a}$$



And then we can change the definition:

$$\frac{|F1F2|}{2a}$$

But this definition isn't perfect, we can still make it cleaner. By using the Pythagorean theorem. Due to the triangle, we constructed earlier being an isosceles triangle. We can cut it in half and get two right-angled triangles. We must find A. C = a because B1 is either on the minor axis or in the middle. This means that the Focal rays are the same length: $(K = |B1F1| + |B1F2| = 2a)$

Which means that:

$$a = |B1F1|$$

Making the right-angled triangle:

$$c^2 \; + b^2 \;\; = a^2$$

$$\to c \mathbin{\hat{}} 2 = a \mathbin{\hat{}} 2 \; - b \mathbin{\hat{}} 2$$

Now we put that into the equation:

$$\frac{2\sqrt{a^2 - b^2}}{2a}$$

And shorten it:

$$\frac{\sqrt{a^2 - b^2}}{a}$$

Then:

$$\frac{a^2 - b^2}{a^2} = e^2$$

Now we take the square root of the it all. We are allowed to do this because the square root of a fraction is the same as the square root of the denominator and the counter divided.

$$\sqrt{\frac{a^2 - b^2}{a^2}} = e$$

a divided by a is one:

$$\sqrt{1 + \frac{-b^2}{a^2}} = e$$

And finally, just a cleanup:

$$\sqrt{1-\left(\frac{b}{a}\right)^2} = e$$

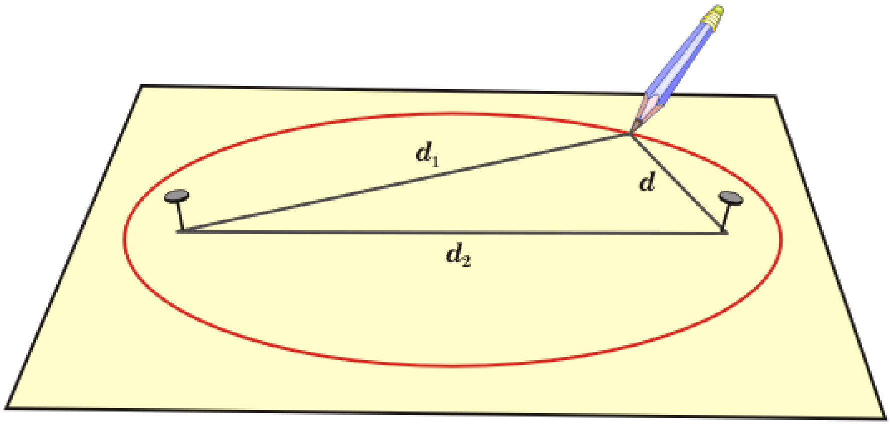
And here is our equation:

$$e = \frac{a}{c} = \sqrt{1-\left(\frac{b}{a}\right)^2 \;\; e = \frac{|F_1F_2|}{|A_1A_2|}}$$

METHODS

The nail and rope method

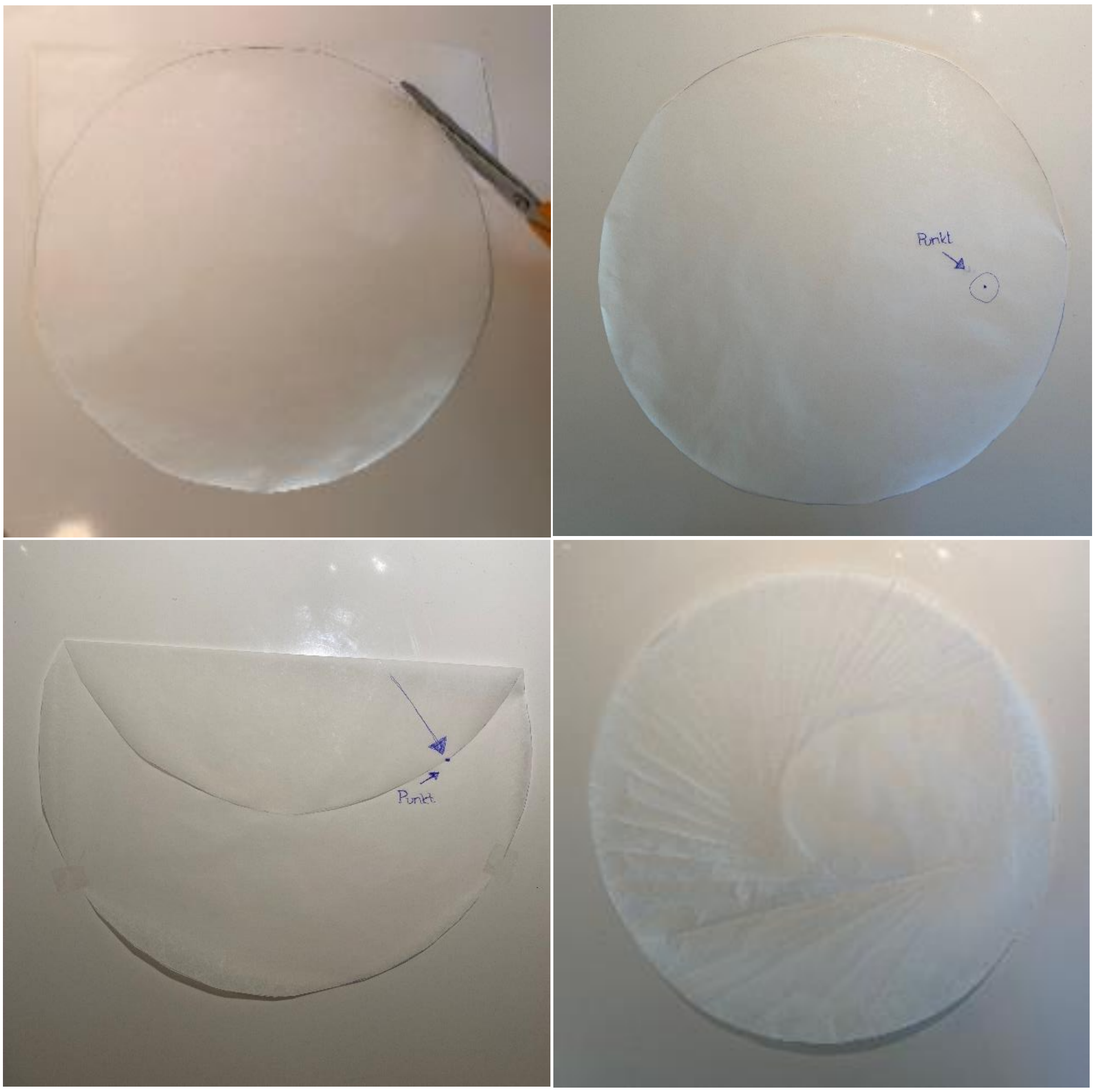
Take a piece of string and tie the ends together so they create a closed loop. Nail two nails into the chosen focal points on a piece of wood. Put the string over the two nails and use a pen to draw on the wood while keeping the string tugged. Then you will see an ellipse appear.



We have also made an example of this method. We have taken a flat piece of wood with dimensions 34x30 cm and made an ellipse using the above method. We took 2 nails and put them on the large axis creating the focal points of the ellipse. Now you too can make an ellipse with string, nails, and a pen. As shown in the illustration.

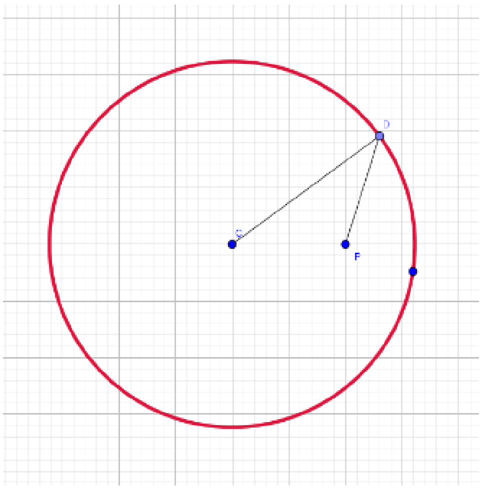
Folding paper to make an ellipse.

First, construct a circle on either paper or parchment and cut it out. Now draw a point (P) on the circle at an arbitrary place on the plane. Then you must fold points on the circle's circumference against the point P (as seen in picture 3 beneath). Continue to fold the circle's circumference all the way around the circle. (See results picture 4)
An ellipse should appear on the paper created by the folds.

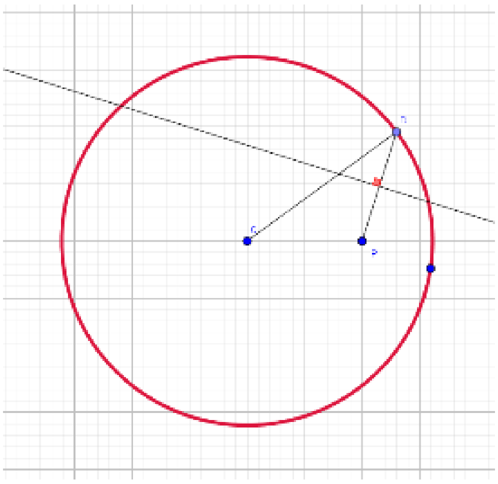


Is this an ellipse or just a flat circle?

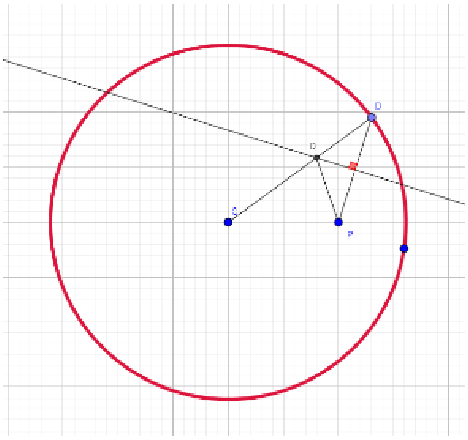
The distinguishing traits of an ellipse is that the sum of the distance between the two focal points to the ellipse is a constant, as mentioned on page 2. In the visual program, GeoGebra (or similar programs) draw a circle. The center of the circle is named C. Now place an arbitrary point P inside the circle. Now choose a point on the circumference of the circle D. Draw the lines between |CD| and |PD|.



Now draw a mid-normal on the line |PD|, which illustrates the folds.



The intersection of the mid-normal and |CD| is called Q. The triangle made from the point P, Q, and D is an isosceles triangle where the lines |PQ| and |QD| have the same length.



The length of the two focal rays | CQ | + | PQ | = | CQ | + |QD| since the sides |PQ| and |QD| have the same length.

|CQ| + |QD| = the circles radius. (See above picture)

If a new D point is chosen, you also get a new Q point. All possible Q points create the circumference of an ellipse since they all fulfill the requirements of an ellipse. Therefore, we can conclude that it is possible to use this method to create an ellipse.

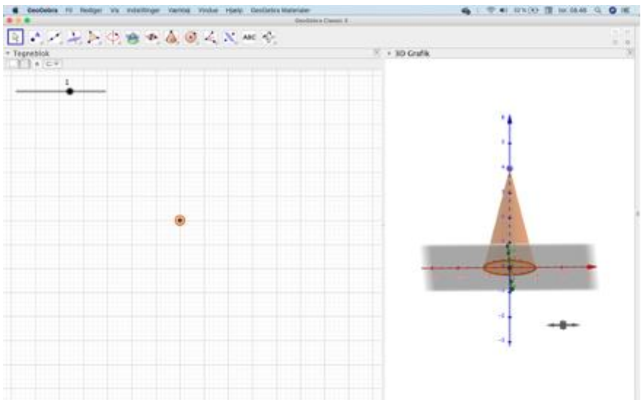
Conic Section

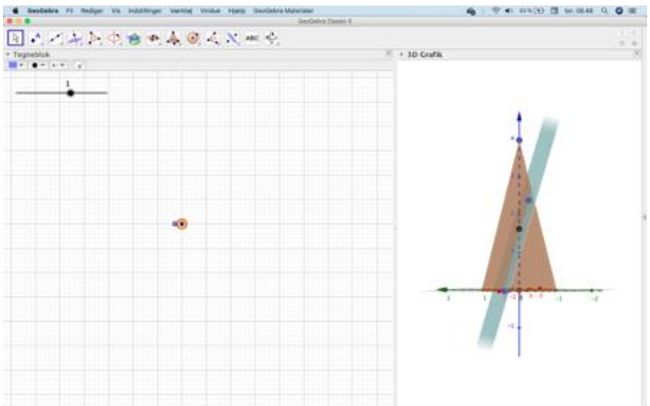
This is how you make a conic section in GeoGebra:

In GeoGebra press the ”show” button and open the 3D plane. Now you can begin.

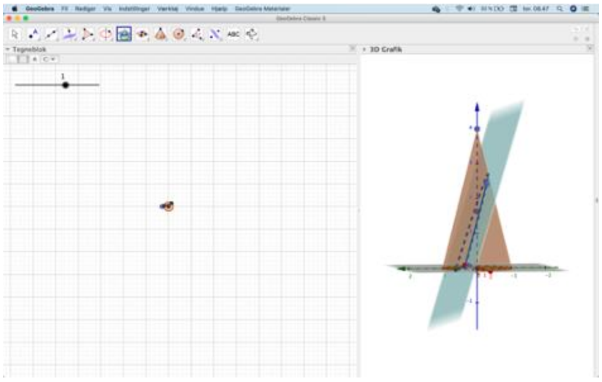
Now press the small arrow beneath the pyramid in the insert menu. Now press the cone option. Press the x or y axis to determine the height of your cone. A menu will now appear prompting you to insert a value for the radius of the cone. Write “a” to create a changeable variable slider for the radius a.

To make a conic section you need to press the ”plane through three points” option in the menu. First, click the side of the cone, then the middle, and finally an arbitrary place outside the cone. You have now made you conic section, but it isn’t an ellipse yet.

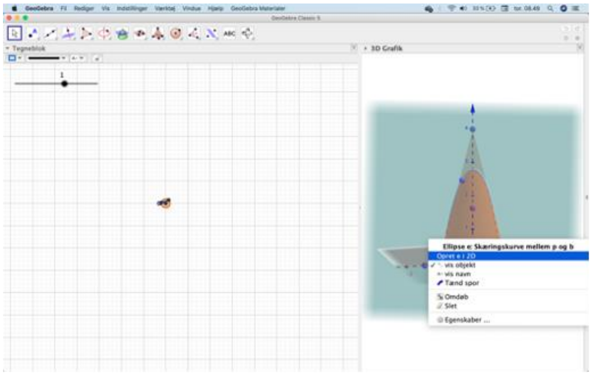




Now you must show your cut, so you must click on the” cut between to planes” tool.



Now hold the mouse a place on the cut you just made, right click, and press the “create in 2D” button to create a 2d menu. Now pull the three points you made on the cone earlier around to create a conic section that creates an ellipse.



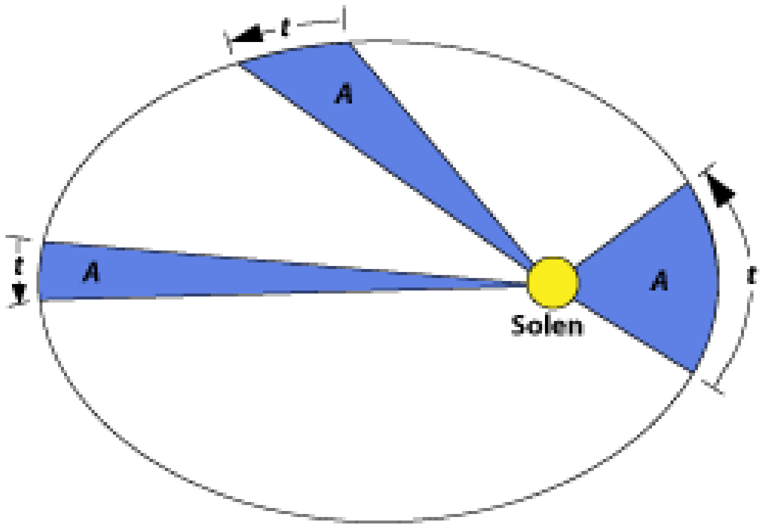
YouTube explanation and demonstration: [Why slicing a cone gives an ellipse]

Kepler

Johanne Kepler was a German mathematician, astronomer, and astrologist. Kepler was born in the year 1571 and enrolled in the University of Tübingen as an 18-year-old. While he was at the university he began to get invested in astronomy. He especially enjoyed the Copernican system. The Copernican system is a theory about the construction of the solar system and the universe with the sun as the center of it with the planets floating around it. In 1596 Kepler wrote his first book where he tried to calculate the difference between the Copernican planet path sizes.

In 1600 Kepler moved to Prague to work with Tycho Brahe who gave him the job to devise a new theory about the planet movement of Mars. After Tycho Brahe died Kepler proved that Mars didn't have an eccentric path, but an "oval" shape, like an ellipse.

Kepler created three rules. His first rule dictated that all planet's paths are shaped like an ellipse, and the ellipse had two focal points, the sun, and a point in space. It also said that the sum of the distance from two focal points to a point on the ellipse was the same all the way around and that the sum was also the same as the length of the major axis.



Kepler's second law described the change in the speed of a planet depending on where it is on its path. A planet moves faster when it is closer to the sun and slower when it's far away. The law described that the area of a given piece of time for example a week on the path around the sun, is the same all the way around. Since the planet's speed was faster close to the sun and slower farther away. Kepler's Third law described the connection between the time it takes to travel the planets path and the length of the planets path. This way he figured out how long time it took the planet to travel all the way around.

[Feynman's Lost Lecture (ft. 3Blue1Brown)]

Equations of the ellipse

Circumference: $2 \cdot \pi \cdot \sqrt{a^2 + \frac{b^2}{2}}$

Area: $\pi \cdot a \cdot b$ a = x axis, b = y axis.

Ellipse formula: $\frac{x^2}{a^2} + \frac{y^3}{b^2} = 1$

a = major axis, b = minor axis (in this case a = x axis and b = the y axis, so the circumscribed square is $2 \cdot a \cdot 2 \cdot b$).

Super-ellipse : $(\frac{x}{a})^n + (\frac{y}{b})^n = 1$

Ellipses in Architecture

Ellipses are also used a bunch in architecture, among others like ellipse-shaped buildings, bridges, and stairs. The Filsø ellipse is an example of the ellipse being used in architecture. It is a bridge and road, located at Filsø in West Jutland, Denmark.

Ellipses in art

The ellipse is a very used shape in art, due to it being a very harmonic shape. We have collected some examples of ellipses used in art (digital art). This art for an example is made from many ellipses which start with its focal points far apart. But as the ellipse spins around itself they grow closer.

Museum of the Future

Museum of the Future is 77 meters tall, and the entire area the statue is in, is 30.548 square meters. The facade of the statue is made of stainless steel, and the statue covers 17.600 square meters. The statue is designed by Killa and built by Buro Happold Engineering consultancy. It was completed the 22. February 2022. The statue is in Dubai, and far away from the tall buildings making the statue the focus point of the area.

The Museum of the Future is a symbol of an ever-evolving future. The ellipse-shaped statue is a symbol of humanity, and the green platform it stands on is representing the planet. What is below must represent the emtiness in the universe. Together, it creates an insecure future.

CONCLUSION

By using our methods to create ellipses we have reached the conclusion that an ellipse is a geometric character defined by its focal rays. During our work we had some hard times getting the whole project started, but when we had figured out who was supposed to do what and split into designated groups. Each group had their own focus points. We were good at communicating and everyone was always updated on what was happening. In the end we put our work together and polished our report. Overall, it was a good working experience for everyone.

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